1. How many leaves does a full 3-ary tree with all leaves at height 4 have?
\[ L = 3^{\text{height}} = 3^4 = 81 \]

2. How many leaves does a 3-ary tree have if it has 15 parents and every parent has exactly 3 children?
There are 15 \cdot 3 + 1 = 46 vertices since every vertex except the root is the child of one of these 15 parents. Thus the number of leaves is 46 - 15 = 31.

3. A full \( m \)-ary tree with \( i \) internal vertices has \( n = mi + 1 \) vertices.

Lecture notes Introduction to Trees, Theorem 4.9.2.

4. An \( m \)-ary tree with height \( h \) has at most \( m^h \) leaves.

Lecture notes Introduction to Trees, Theorem 4.11.1.

5. Find the prefix and postfix forms for the algebraic expressions \((a \ast b) + (c / (d \uparrow 3))\) and \((a \ast ((b + c) / d \uparrow 3))\).

**Prefix:** + * a b / c \uparrow d 3; * a \uparrow / + b c d 3

**Postfix:** a b * c d 3 \uparrow / +; a b c + d / 3 \uparrow *

6. What is the value of the prefix expression: \(+ - \uparrow 3 2 \uparrow 2 3 / 6 - 4 2 = 4\)

7. What is the value of the postfix expression: \(9 3 / 5 + 7 2 - * = 40\)

8. How many Boolean functions of order \( n \) are there?

\(2^n\)

9. Define the Boolean function, \( F \), in the three variables, \( x, y, \) and \( z \), by \( F(1, 1, 0) = F(1, 0, 1) = F(0, 0, 0) = 1\) and \( F(x, y, z) = 0\) for all other \((x, y, z)\) in \(\{0, 1\}^3\).

(a) Find the sum-of-products form for \( F \).

\[ F(x, y, z) = xy\bar{z} + x\bar{y}z + \bar{x}y\bar{z} \]

(b) Find the product-of-sums form for \( F \).

\[ F(x, y, z) = (x + y + z)(x + y + \bar{z})(\bar{x} + y + z)(x + \bar{y} + z)(x + y + \bar{z}) \]

(c) Find the dual of the expression in 9a.

\[ F^d(x, y, z) = (x + y + \bar{z})(x + \bar{y} + z)(\bar{x} + \bar{y} + \bar{z}) \]

10. Prove or disprove the set is functionally complete.

(a) \{+\} is not functionally complete. \(0 + 0 = 0\), so + cannot create an output of 1 out of an input of all 0’s.

(b) \{\cdot\} is not functionally complete. \(0 \cdot 0 = 0\), so \cdot cannot create an output of 1 out of an input of all 0’s.

(c) \{-\} is not functionally complete. This is a unary operator and cannot combine 2 or more values.
(d) \{+, \cdot\} is not functionally complete. \(0 \cdot 0 = 0\) and \(0 + 0 = 0\), so no combination of \(\cdot\) and \(+\) can create an output of 1 out of an input of all 0’s.

(e) \{+, −\} is functionally complete. The only operator of the known functionally complete set \{+, \cdot, −\} that is missing is \(\cdot\). By DeMorgan’s Laws we have

\[x \cdot y = \overline{x + y}\]

(f) \{−\} is functionally complete. The only operator of the known functionally complete set \{+, \cdot, −\} that is missing is \(+\). By DeMorgan’s Laws we have

\[x + y = \overline{x \cdot y}\]

(g) \{\downarrow\} is functionally complete. Use exercise 15 on page 712 in the text.

(h) \{||\} is functionally complete. Use exercise 14 on page 712 in the text.

11. Prove or disprove the stated equivalence, where \(x, y\) and \(z\) are variables of a Boolean function.

(a) \(x \downarrow y = \overline{x y}\).

\[
\begin{array}{c|cc|cc|cc}
\hline
x & y & x \downarrow y & \overline{x} & y & x \downarrow y & \overline{x} y \\
\hline
1 & 1 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 \\
\hline
\end{array}
\]

Since both \(x \downarrow y = \overline{x y}\) for every \((x, y) \in \{0, 1\}^2\), the expressions are equivalent.

(b) \((x + 1) = (x + 1)(x + 1)\)

\[
\begin{array}{c|c|c|c|c}
\hline
x & x + 1 & x x & (x + 1)(x + 1) & x x + 1 \\
\hline
1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 \\
\hline
\end{array}
\]

Since both \((x + 1)\) and \((x + 1)(x + 1)\) for every \(x \in \{0, 1\}\), the expressions are equivalent.

(c) \(yz + \overline{x} = \overline{yzx}\)

\[
\begin{array}{c|c|c|c|c|c|c|c}
\hline
x & y & z & \overline{x} & yz & yz + \overline{x} & \overline{yz} & \overline{yzx} \\
\hline
1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
\hline
\end{array}
\]

Since both \(yz + \overline{x}\) and \(\overline{yzx}\) for every \((x, y, z) \in \{0, 1\}^3\), the expressions are equivalent.
12. Prove the law holds for an abstract Boolean algebra. Use only the properties of a Boolean algebra and the previously stated laws.

(a) Idempotent Laws: \( x + x = x \) and \( x \cdot x = x \)
One is in Lecture Notes Abstract Boolean Algebras section 5.

(b) Dominance Laws: \( x + 1 = 1 \) and \( x \cdot 0 = 0 \)
One is in Lecture Notes Abstract Boolean Algebras section 6.

(c) Absorption Laws: \((x \cdot y) + x = x\) and \((x + y) \cdot x = x\)

Proof of \((x \cdot y) + x = x\).
\[
(x \cdot y) + x = (x \cdot y) + (x \cdot 1) \quad \text{by the Identity Laws}
\]
\[
= x(y + 1) \quad \text{by the distributive laws}
\]
\[
= x(1) \quad \text{by the previous laws}
\]
\[
= x \quad \text{by the identity laws}
\]
\[\square\]

Proof of \((x + y) \cdot x = x\). The dual of \((x + y) \cdot x = x\) is \((x \cdot y) + x = x\). Since we know the latter equivalence is true, the first expression must also be valid by the principal of duality.
\[\square\]

(d) \(w + z = 1\) and \(w \cdot z = 0\) if and only if \(z = \overline{w}\)
Lecture Notes Abstract Boolean Algebras section 7.

(e) Double Compliments Law: \(\overline{x} = x\)

Proof. \(\overline{x} = \overline{x} \cdot 1\) Identity laws
\[
= \overline{x}(x + x) \quad \text{Compliments Laws}
\]
\[
= \overline{x}x + \overline{x}x \quad \text{Distributive Laws}
\]
\[
= 0 + \overline{x}x \quad \text{Compliments Laws}
\]
\[
= \overline{x}x + \overline{x}x \quad \text{Compliments Laws}
\]
\[
= (\overline{x} + \overline{x}) \cdot x \quad \text{Distributive Laws}
\]
\[
= 1 \cdot x \quad \text{Compliments Laws}
\]
\[
= x \quad \text{Identity Laws}
\]
\[\square\]

(f) DeMorgan’s Laws: \(\overline{x \cdot y} = \overline{x} + \overline{y}\) and \(\overline{x + y} = \overline{x} \cdot \overline{y}\)
One is in Lecture Notes Abstract Boolean Algebras section 8.

13. Let \(B\) be a Boolean algebra and let \(\preceq\) be the relation on \(B\) given by \(x \preceq y\) if and only if \(x + y = y\). Prove \(\preceq\) is a partial order. You may only use the definition of a Boolean algebra and the definition of the relation \(\preceq\).

Proof. Reflexive: Let \(x \in B\).
\[
x + x = (x + x) \cdot 1 \quad \text{by the Identity Laws},
\]
\[
= (x + x)(x + \overline{x}) \quad \text{by the Compliments Laws},
\]
\[
= x + (x \cdot \overline{x}) \quad \text{by the Distributive Laws},
\]
\[
= x + 0 \quad \text{by the Compliments Laws},
\]
\[
= x \quad \text{by the Identity Laws}.
\]
(reproves the idempotent property)
Thus \(x \preceq x\). Since \(x \in B\) was arbitrary, this shows \(\preceq\) is reflexive.
Antisymmetric: Let \( x, y \in B \) be such that \( x \preceq y \) and \( y \preceq x \). Then by the definition of \( \preceq \) we have \( x + y = x \) and \( y + x = y \). By the commutative laws we have \( x + y = y + x \), so \( x = y \). This shows \( \preceq \) is antisymmetric.

Transitive: Let \( x, y, z \in B \) be such that \( x \preceq y \) and \( y \preceq z \). Then by definition of \( \preceq \) we have \( x + y = x \) and \( y + z = y \). Thus
\[
x + z = (x + y) + z \text{ since } x + y = x
\]
\[
= x + (y + z) \text{ by associative laws}
\]
\[
= x + y \text{ since } y + z = y
\]
\[
= x \text{ by } x + y = x.
\]
This shows \( x \preceq z \) and so \( \preceq \) is transitive.

\( \square \)

\( \forall x \in B_1[\phi(ax) = \phi(0) = 0 \text{ or } \phi(ax) = \phi(a)]. \) Since \( \phi \) is an isomorphism this is true if and only if \( \forall x \in B_1[\phi(a)\phi(x) = 0 \text{ or } \phi(a)\phi(x) = \phi(a)]. \)

\( \forall y \in B_2[\phi(a)y = 0 \text{ or } \phi(a)y = \phi(a)]. \)

14. Let \( \mathcal{B} \) be an abstract Boolean algebra. Prove a nonzero element \( a \in \mathcal{B} \) is an atom if and only if for every \( x, y \in \mathcal{B} \) with \( a = x + y \) we must have \( a = x \) or \( a = y \).

This is Theorem 3.10.1 in Abstract Boolean Algebras.

15. Let \( B_1 \) and \( B_2 \) be Boolean algebras and let \( \phi : B_1 \to B_2 \) be a Boolean algebra isomorphism. Prove \( a \preceq b \) in \( B_1 \) if and only if \( \phi(a) \preceq \phi(b) \) in \( B_2 \) where \( \preceq \) is the partial order on a Boolean algebra defined in 13.

Proof. Let \( a, b \in B_1 \).
\[
a \preceq b
\]
\[
\iff a + b = b
\]
\[
\iff \phi(a + b) = \phi(b) \text{ since } \phi \text{ is a bijection}
\]
\[
\iff \phi(a) + \phi(b) = \phi(b) \text{ since } \phi \text{ preserves Boolean operations}
\]
\[
\iff \phi(a) \preceq \phi(b).
\]

\( \square \)

16. Provide the rest of the information in the table.

(a) \( B = \{0, 1\} \)
\[
0 \text{ element: } 0
\]
\[
1 \text{ element: } 1
\]
\[
an \text{ element that is neither 0 nor 1 nor atom: } \text{none}
\]

(b) \( B^5 \)
\[
0 \text{ element: } (0, 0, 0, 0, 0)
\]
\[
1 \text{ element: } (1, 1, 1, 1, 1)
\]
\[
an \text{ element that is neither 0 nor 1 nor atom: } (1, 1, 1, 0, 0)
\]

(c) \( BOOL(2) \)
\[
0 \text{ element: } f_0 : B^2 \to B \text{ defined by } f_0(x, y) = 0 \text{ for all } (x, y) \in B^2
\]
\[
1 \text{ element: } f_1 : B^2 \to B \text{ defined by } f_1(x, y) = 1 \text{ for all } (x, y) \in B^2
\]
\[
an \text{ element that is neither 0 nor 1 nor atom: } f_2 : B^2 \to B \text{ defined by } f_2(1, 0) = f_2(1, 1) = 1 \text{ and } f_2(0, 0) = f_2(0, 1) = 0.
\]

(d) \( P(\{a, b, c, d\}) \)
\[
0 \text{ element: } \emptyset
\]
1 element: \{a, b, c, d\}
an element that is neither 0 nor 1 nor atom: \{a, b\}

(e) \(FUN(\{a, b, c\}, B)\)

0 element: \(g_0: \{a, b, c\} \to B\) defined by \(g_0(x) = 0\) for all \(x \in \{a, b, c\}\)

1 element: \(g_1: \{a, b, c\} \to B\) defined by \(g_1(x) = 1\) for all \(x \in \{a, b, c\}\)
an element that is neither 0 nor 1 nor atom: \(g_2: \{a, b, c\} \to B\) defined by \(g_2(a) = g_2(b) = 1\) and \(g_2(c) = 0\)

(f) \(D_6\)

0 element: 1
1 element: 6
an element that is neither 0 nor 1 nor atom: none

17. List the atoms of each of the Boolean algebras in the table in exercise 16.

(a) \(B = \{0, 1\}\)
atoms: 1

(b) \(B^5\)
atoms: (1,0,0,0,0), (0,1,0,0,0), (0,0,1,0,0), (0,0,0,1,0), (0,0,0,0,1)

(c) \(BOOL(2)\) atoms:
\(f_3: B^2 \to B\) defined by \(f_3(1, 0) = 1\) and \(f_3(x, y) = 0\) for all other \((x, y) \in B^2\)
\(f_4: B^2 \to B\) defined by \(f_4(0, 1) = 1\) and \(f_4(x, y) = 0\) for all other \((x, y) \in B^2\)
\(f_5: B^2 \to B\) defined by \(f_5(0, 0) = 1\) and \(f_5(x, y) = 0\) for all other \((x, y) \in B^2\)
\(f_6: B^2 \to B\) defined by \(f_6(1, 1) = 1\) and \(f_6(x, y) = 0\) for all other \((x, y) \in B^2\)

(d) \(P(\{a, b, c, d\})\)
atoms: \{a\}, \{b\}, \{c\}, \{d\}

(e) \(FUN(\{a, b, c\}, B)\)
atoms
\(g_3: \{a, b, c\} \to B\) defined by \(g_3(a) = g_3(b) = 0\) and \(g_3(c) = 1\)
\(g_4: \{a, b, c\} \to B\) defined by \(g_4(a) = g_4(c) = 0\) and \(g_4(b) = 1\)
\(g_5: \{a, b, c\} \to B\) defined by \(g_5(c) = g_5(b) = 0\) and \(g_5(a) = 1\)

(f) \(D_6\)
atoms: 2, 3

18. Express each element given in the 4th column in table 16 as a sum of atoms. Note that your answer depends on your answer to 16 and 17.

(a) \(B = \{0, 1\}\) none

(b) \((1, 1, 1, 0, 0) = (1, 0, 0, 0, 0) + (0, 1, 0, 0, 0) + (0, 0, 1, 0, 0)\)

(c) \(f_2 = f_3 + f_6\)

(d) \(\{a, b\} = \{a\} + \{b\}\)

(e) \(g_2 = g_4 + g_5\)

(f) \(D_6\) none

19. Draw a Hasse diagram for each of the Boolean algebras in the table in exercise 16 with \(S = \{a, b, c\}\) and the partial order \(\preceq\) defined in exercise 13.
The Characteristic function is used here to define the functions.
20. How many elements does a finite Boolean algebra have if the number of atoms is 5?

32

21. Define an isomorphism between $FUN(\{a, b, c\}, B)$ and $B^3$. Answer is not unique.

$f : FUN(\{a, b, c\}, B) \rightarrow B^3$ is defined by

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\chi_{\emptyset}$</th>
<th>$\chi_{{a}}$</th>
<th>$\chi_{{b}}$</th>
<th>$\chi_{{c}}$</th>
<th>$\chi_{{a,b}}$</th>
<th>$\chi_{{a,c}}$</th>
<th>$\chi_{{b,c}}$</th>
<th>$\chi_{{a,b,c}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>(0, 0, 0)</td>
<td>(1, 0, 0)</td>
<td>(0, 1, 0)</td>
<td>(0, 0, 1)</td>
<td>(1, 1, 0)</td>
<td>(1, 0, 1)</td>
<td>(0, 1, 1)</td>
<td>(1, 1, 1)</td>
</tr>
</tbody>
</table>