ALGEBRA QUALIFYING EXAM

AUGUST 24, 2001 — 1:00–5:00 PM

Please attempt all six problems.

(1) Let $G$ be a group with normal subgroups of order 3, resp. 5. Show that $G$ has an element of order 15.

(2) Let $G$ be an abelian group.
   (i) Prove that if $G$ is finitely generated, then $G/2G$ is a finite group.
   (ii) Give an example showing that $G$ is not necessarily finitely generated even if $G/2G$ is a finite group.

(3) Let $R$ be a commutative ring with 1, and let $a$ be an ideal of $R[x]$. Let $n$ be the smallest degree of a nonzero element in $a$, and assume that $a$ contains a monic polynomial of degree $n$. Prove that $a$ is principal.

(4) (i) Prove that if $a$, $b$, and $c$ are elements of a finite field $F$, and $c = ab$, then at least one of $a$, $b$, or $c$ is a square.
   (ii) Is this fact still necessarily true over an infinite field? (proof or counterexample)

(5) Let $M$ be a module over the commutative unitary ring $R$, and let $N$ be a submodule of $M$.
   (i) Prove that if $N$ and $M/N$ are finitely generated, then so is $M$.
   (ii) Prove or disprove the converse of (i).

(6) Let $\mathbb{Z}[x]$ be the ring of polynomials in the indeterminate $x$ over the ring $\mathbb{Z}$ of integers. Let $D$ be the subset of $\mathbb{Z}[x]$ consisting of all polynomials with coefficient of $x$ equal to 0. You may assume without proof that $D$ is an integral domain.
   (i) Prove that $D$ is not a UFD.
   (ii) Prove that the ideal $(x^2, x^3)$ is prime, but not maximal, in $D$. 