ALGEBRA QUALIFYING EXAM

JANUARY 7, 2000 — 9:00 AM–1:00 PM

Please attempt all six problems.

(1) An abelian group $G$ is ‘divisible’ if $\forall g \in G$ and $\forall n \in \mathbb{N}$, $n > 0$, $\exists h \in G$ such that $nh = g$. Show that no non-trivial finitely generated abelian group is divisible.

(2) Show explicitly that $(\mathbb{Z} \times \mathbb{Z})/\langle (5, 12) \rangle$ is a cyclic group.

(3) Show that every group of order 176 is solvable.

(4) Let $R$ be a commutative ring with 1. Assume that for each $a \in R$ there is an integer $n > 1$ such that $a^n = a$. Prove that every prime ideal of $R$ is a maximal ideal.

(5) Show that every $2 \times 2$ matrix over $\mathbb{R}$ with negative determinant is diagonalizable over $\mathbb{R}$.

(6) Prove that an $m \times n$ matrix $A$ has rank at most 1 if and only if it can be written as $A = BC$ where $B$ is an $m \times 1$ matrix, and $C$ is a $1 \times n$ matrix.