

7. ROTATIONS

This lecture is a more formal discussion of orthogonal transformations, the orthogonal group $O(3)$, and especially the orientation preserving ones, $SO(3)$, the rotations.

A matrix is defined to be orthogonal if the entries are real and

$$(1) \quad \mathbf{A}' \mathbf{A} = \mathbf{I}.$$

Condition (1) says that the gram matrix of the sequence of vectors formed by the columns of \mathbf{A} is the identity, so the columns are an orthonormal frame.

Condition (1) also shows that \mathbf{A} is a rigid motion. A matrix satisfying (1) preserves the dot product since

$$\begin{aligned} \mathbf{A}\mathbf{v} \cdot \mathbf{A}\mathbf{w} &= (\mathbf{A}\mathbf{v})' \mathbf{A}\mathbf{w} = \mathbf{v}' \mathbf{A}' \mathbf{A}\mathbf{w} \\ &= \mathbf{v}' \mathbf{I} \mathbf{w} = \mathbf{v}' \mathbf{w} = \mathbf{v} \cdot \mathbf{w}. \end{aligned}$$

Thus the dot product of two vectors is the same before and after an orthogonal transformation. Lengths and angles are also preserved, since they can be written in terms of dot products.

Orthogonal transformations have determinant ± 1 since by (1) and properties of determinant,

$$\begin{aligned} (\det \mathbf{A})^2 &= \det(\mathbf{A}') \det \mathbf{A} \\ &= \det(\mathbf{A}' \mathbf{A}) \\ &= \det \mathbf{I} = 1. \end{aligned}$$

7.1. The rotation group. If we think of an orthogonal matrix \mathbf{A} as a frame

$$\mathbf{A} = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3),$$

then the determinant is the scalar triple product

$$\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3).$$

The frame is right handed if the triple product is 1 and left handed if it is -1. The frame is the image of the right handed standard frame

$$(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = \mathbf{I}$$

under the rotation given by left multiplication by \mathbf{A} . Thus \mathbf{A} preserves orientation if the determinant is 1. The rotation group, denoted $SO(3)$, consists of all orthogonal transformations with determinant 1.

In two dimensions, every rotation is of the form

$$(2) \quad \mathbf{R}(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

Note that

$$\mathbf{R}(\theta)\mathbf{R}(\phi) = \mathbf{R}(\theta + \phi) = \mathbf{R}(\phi)\mathbf{R}(\theta),$$

so that rotations in two dimension commute. Rotations in dimension 3 do not commute.

Matrices for rotation about coordinate axes have a form related to the 2 dimensional rotation matrices:

Rotation about the x axis

$$\mathbf{R}_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix}$$

Rotation about the y axis

$$\mathbf{R}_y(\theta) = \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix}$$

Rotation about the z axis

$$\mathbf{R}_z(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The rotations $\mathbf{R}_z(\theta)$, are exactly the ones that leave the vector $\mathbf{e}_3 = (0, 0, 1)'$ fixed and can be identified with rotation in the xy plane. Similarly for \mathbf{R}_x and \mathbf{R}_y . The rotations $\mathbf{R}_x(\theta)$ commute,

$$\begin{aligned} \mathbf{R}_x(\theta)\mathbf{R}_x(\phi) &= \mathbf{R}_x(\theta + \phi) \\ &= \mathbf{R}_x(\phi)\mathbf{R}_x(\theta). \end{aligned}$$

Similarly rotations $\mathbf{R}_y(\theta)$ commute and rotations $\mathbf{R}_z(\theta)$ commute. In general, however, rotations in three dimensions do not commute. For example,

$$\mathbf{R}_x(\pi)\mathbf{R}_z(\pi/2) = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

but

$$\mathbf{R}_z(\pi/2)\mathbf{R}_x(\pi) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Since $\text{SO}(3)$ is a group we can get other rotations by multiplying together rotations of the form \mathbf{R}_x , \mathbf{R}_y , and \mathbf{R}_z . It can be shown that any rotation \mathbf{A} can be written as a product of three rotations about the y and z axes,

$$(3) \quad \mathbf{A} = \mathbf{R}_z(\alpha)\mathbf{R}_y(\gamma)\mathbf{R}_z(\delta).$$

The angles α , γ , δ are called *Euler angles* for the rotation \mathbf{A} .

7.2. Rotations and cross products. Rotations are orthogonal transformations which preserve orientation. This is equivalent to the fact that they preserve the vector cross product:

$$(4) \quad \mathbf{A}(\mathbf{v} \times \mathbf{w}) = \mathbf{A}\mathbf{v} \times \mathbf{A}\mathbf{w},$$

for all \mathbf{A} in $\text{SO}(3)$ and \mathbf{v} and \mathbf{w} in \mathbb{R}^3 . Recall the right hand rule in the definition of the cross product (fig 1). The cross product is defined in terms of lengths and angles and right-handedness, all of which remain unchanged after rotation.

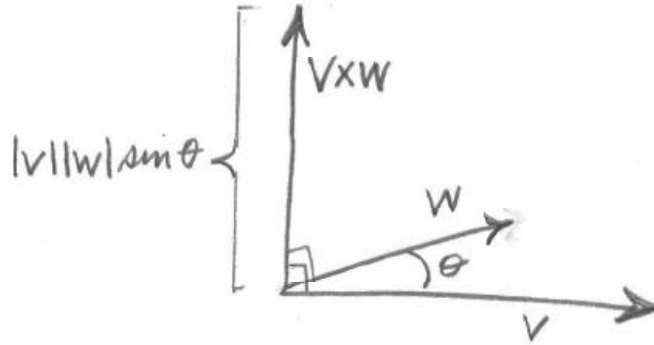


FIGURE 1. The geometric definition of the cross product. The direction of $\mathbf{v} \times \mathbf{w}$ is determined by the right hand rule. The fingers of the right hand should point from \mathbf{v} to \mathbf{w} and the thumb in the direction of the cross-product. The length of the cross product is $|\mathbf{v}||\mathbf{w}|\sin\theta$.

7.3. **Complex form of a rotation.** In dimension 2 it is convenient to use complex numbers to write rotations. Rotation by an angle θ is given by

$$(5) \quad \zeta \rightarrow e^{i\theta}\zeta$$

where $\zeta = x + iy$. A similar method works in three dimensions for rotation about the z axis. First perform a change of coordinates. Write

$$\zeta = \frac{1}{\sqrt{2}}(x + iy)$$

so that $x = \frac{1}{\sqrt{2}}(\zeta + \bar{\zeta})$ and $y = \frac{-i}{\sqrt{2}}(\zeta - \bar{\zeta})$.

Expressing the rotation $R_z(\theta)$ in terms of the new coordinates ζ and we have in addition to (5), $\bar{\zeta} \rightarrow e^{-i\theta}\bar{\zeta}$ and $z \rightarrow z$.

To express this in matrix form let

$$(6) \quad \mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \mathbf{Z} = \begin{pmatrix} \zeta \\ \bar{\zeta} \\ z \end{pmatrix}$$

and define the matrices

$$(7) \quad \mathbf{D}(\theta) = \begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{-i\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-i}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The eigenvalues of $R_z(\theta)$ are the diagonal entries of $D(\theta)$ and corresponding eigenvectors are given by columns of \mathbf{B} . In matrix form,

$$(8) \quad R_z(\theta)\mathbf{B} = \mathbf{B}D(\theta).$$

Note that \mathbf{B} is self-adjoint,

$$(9) \quad \mathbf{B}^*\mathbf{B} = I.$$

This means the eigenvectors are normalized: their hermitian length is 1 and they are hermitian orthogonal.

Now $X = \mathbf{B}Z$ and so \mathbf{B} is a frame with complex vectors and Z gives the coordinates of the vector X in the frame \mathbf{B} .

Rotation an angle θ about the z axis is given in the new coordinates by $Z \rightarrow D(\theta)Z$. So using the coordinates $(\zeta, \bar{\zeta}, z)$ in 3D space, the matrix for rotation about the z axis is diagonal.

7.3.1. Hermitian geometry. The use of vectors and matrices with complex entries can be explained using some definitions from hermitian geometry.

For vectors \mathbf{v} and \mathbf{w} , the *hermitian inner product* is defined by

$$\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^* \mathbf{w}.$$

If \mathbf{v} and \mathbf{w} are real vectors the hermitian inner product is the same as the dot product. Two vectors are *hermitian orthogonal* if their hermitian inner product is 0. The length $\|\mathbf{v}\|$ of a vector \mathbf{v} is defined by

$$\|\mathbf{v}\|^2 = \langle \mathbf{v}, \mathbf{v} \rangle.$$

For real vectors, this is the same as the usual definition.

A square matrix \mathbf{A} is said to be *unitary* if

$$\mathbf{A}^* \mathbf{A} = \mathbf{I}.$$

If the entries of \mathbf{A} are real then unitary is the same as orthogonal. The matrix $\mathbf{A}^* \mathbf{A}$ is the gram matrix of hermitian inner products of the vectors forming the columns of \mathbf{A} .

Note that in equation (7) the matrices \mathbf{B} and $\mathbf{D}(\theta)$ are unitary and since $X = \mathbf{B}Z$, the vectors X and Z have the same length.

The following is left as an exercise:

If \mathbf{v} and \mathbf{w} are eigenvectors of a unitary matrix corresponding to distinct eigenvalues then $\mathbf{v}^ \mathbf{w} = 0$.*

7.4. Eigenvalues of a rotation. By choosing the correct coordinate frame, any rotation has the matrix form $D(\theta)$. This follows from the fact that

Theorem: The eigenvalues of a rotation are of the form

$$(10) \quad e^{i\theta}, e^{-i\theta}, 1$$

for some angle θ . There are corresponding eigenvectors of the form

$$(11) \quad (\mathbf{v}, \bar{\mathbf{v}}, \mathbf{u})$$

where \mathbf{u} is a vector with real coordinates and where $(\mathbf{v}, \bar{\mathbf{v}}, \mathbf{u})$ is unitary with determinant 1.

It follows that θ is the angle of rotation and the eigenvector \mathbf{u} corresponding to the eigenvalue 1 is along the axis of the rotation.

First we show that *the eigenvalues of an orthogonal matrix have absolute value 1*.

To see this, suppose

$$(12) \quad \mathbf{A}\mathbf{v} = \lambda\mathbf{v}$$

for a non-zero vector \mathbf{v} . Taking the adjoint,

$$\mathbf{v}^*\mathbf{A}^* = \bar{\lambda}\mathbf{v}^*.$$

Since \mathbf{A} is real, $\mathbf{A}^* = \mathbf{A}'$, and multiplying (12) and (7.4) and using the fact that $\mathbf{A}'\mathbf{A} = \mathbf{I}$ get

$$\mathbf{v}^*\mathbf{v} = |\lambda|^2\mathbf{v}^*\mathbf{v}$$

and hence

$$|\lambda|^2 = 1.$$

Since \mathbf{A} is real, if λ is an eigenvalue, so is $\bar{\lambda}$ and if \mathbf{v} is an eigenvector corresponding to λ , then $\bar{\mathbf{v}}$ is an eigenvector corresponding to $\bar{\lambda}$. It follows that the eigenvalues of an orthogonal matrix A are

$$\pm 1 \quad \lambda \quad \bar{\lambda}$$

where $\lambda = e^{i\theta}$. The determinant of \mathbf{A} is the product of the eigenvalues, so for a rotation matrix the first eigenvalue above is 1. There is a real eigenvector \mathbf{u} corresponding to the eigenvalue 1. This is left as an exercise.

Now we show that $(\mathbf{v}, \bar{\mathbf{v}}, \mathbf{u})$ can be chosen to be unitary. If $e^{i\theta} \neq \pm 1$ then the eigenvalues $e^{i\theta}, e^{-i\theta}$ are distinct and \mathbf{v} and $\bar{\mathbf{v}}$ are orthogonal. Dividing the vectors \mathbf{v} and \mathbf{u} by their length normalizes so that $(\mathbf{v}, \bar{\mathbf{v}}, \mathbf{u})$ is unitary. If the determinant of $(\mathbf{v}, \bar{\mathbf{v}}, \mathbf{u})$ is -1 then replace θ by $-\theta$ and exchange \mathbf{v} and $\bar{\mathbf{v}}$. Now the determinant is 1.

If $e^{i\theta} = \pm 1$, the eigenvalues $e^{i\theta}, e^{-i\theta}$ are not distinct, but the result holds. This is left as an exercise.

Note that the result of the theorem can be written

$$(13) \quad \mathbf{A}\mathbf{F} = \mathbf{F}\mathbf{D}(\theta) \quad \mathbf{F} = (\mathbf{v}, \bar{\mathbf{v}}, \mathbf{u}) \quad \mathbf{F}^*\mathbf{F} = \mathbf{I} \quad \det \mathbf{F} = 1$$

where $\mathbf{D}(\theta)$ is as in (7).

7.4.1. Axis and angle of rotation. A vector \mathbf{u} which is an eigenvector corresponding to the eigenvalue 1 is called an axis direction for \mathbf{A} . The vector \mathbf{u} is fixed by \mathbf{A} since $\mathbf{A}\mathbf{u} = \mathbf{u}$. The angle θ is called the angle of rotation about \mathbf{u} . Write

$$(14) \quad \mathbf{A} = \mathbf{R}(\mathbf{u}, \theta)$$

indicating that the rotation is an angle θ counterclockwise about an axis in the direction \mathbf{u} . The angle is not unique since

$$(15) \quad \mathbf{R}(\mathbf{u}, \theta) = \mathbf{R}(-\mathbf{u}, -\theta)$$

but the absolute value of the angle is uniquely determined. The axis line, but not the direction of the line, is uniquely determined.

7.5. Properties of rotations. A few of the main properties of rotations are summarized here. In what follows, \mathbf{u} is a unit vector and \mathbf{v} and \mathbf{w} are real vectors, and \mathbf{A} is a rotation,

$$(16) \quad \mathbf{R}(\mathbf{u}, \theta) = \mathbf{R}(-\mathbf{u}, -\theta)$$

$$(17) \quad \mathbf{A}\mathbf{v} \cdot \mathbf{A}\mathbf{w} = \mathbf{v} \cdot \mathbf{w}$$

$$(18) \quad \mathbf{A}\mathbf{v} \times \mathbf{A}\mathbf{w} = \mathbf{A}(\mathbf{v} \times \mathbf{w})$$

$$(19) \quad \mathbf{u} \cdot (\mathbf{R}(\mathbf{u}, \theta)\mathbf{v} - \mathbf{v}) = \mathbf{0}$$

$$(20) \quad \mathbf{A}\mathbf{R}(\mathbf{u}, \theta)\mathbf{A}^{-1} = \mathbf{R}(\mathbf{A}\mathbf{u}, \theta)$$

$$(21) \quad \mathbf{R}(\mathbf{u}, \theta)\mathbf{v} = \cos \theta (\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{u}) \\ + \sin \theta \mathbf{u} \times \mathbf{v} + (\mathbf{u} \cdot \mathbf{v})\mathbf{u}$$

The identities (16), (17) and (18) were shown above.

Proof of (19). Let $\mathbf{A} = \mathbf{R}(\mathbf{u}, \theta)$. Since $\mathbf{A}\mathbf{u} = \mathbf{u}$,

$$\begin{aligned} \mathbf{u} \cdot (\mathbf{A}\mathbf{v} - \mathbf{v}) &= \mathbf{u} \cdot \mathbf{A}\mathbf{v} - \mathbf{u} \cdot \mathbf{v} \\ &= \mathbf{A}\mathbf{u} \cdot \mathbf{A}\mathbf{v} - \mathbf{u} \cdot \mathbf{v} \\ &= \mathbf{0}. \end{aligned}$$

The last equality follows from (17). \square

Proof of (20). Check that the eigenvalues of the matrices on either side of the equation are $e^{i\theta}$, $e^{-i\theta}$, 1 corresponding to eigenvectors $\mathbf{A}\mathbf{v}$, $\mathbf{A}\bar{\mathbf{v}}$, $\mathbf{A}\mathbf{u}$. \square

Proof of (21). Multiplying both sides of the equation by a rotation \mathbf{A} and using (17), (18), and (20) gives

$$\mathbf{R}(\mathbf{A}\mathbf{u}, \theta)\mathbf{A}\mathbf{v} = \cos \theta (\mathbf{A}\mathbf{v} - (\mathbf{A}\mathbf{u} \cdot \mathbf{A}\mathbf{v})\mathbf{A}\mathbf{u})$$

so without loss of generality assume $\mathbf{u} = \mathbf{e}_3$ in eqrefrotvect. Now check that the equation holds for $\mathbf{v} = \mathbf{e}_1$ and $\mathbf{v} = \mathbf{e}_2$.

The formula can also be seen geometrically. The vector \mathbf{v} is written as the sum of its projection onto \mathbf{u} and onto the plane perpendicular to \mathbf{u} ,

$$\mathbf{v} = (\mathbf{v} \cdot \mathbf{u})\mathbf{u} + (\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{u}).$$

The projection onto \mathbf{u} is left fixed and the projection onto the plane perpendicular to \mathbf{u} is rotated an angle θ in the plane. \square

Equation (21) gives a way to compute the matrix $\mathbf{R}(\mathbf{u}, \theta)$ for a unit vector \mathbf{u} . If $\mathbf{u} = (a, b, c)'$ write

$$(22) \quad \mathbf{S}_{\mathbf{u}} = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$$

Note that for any vector \mathbf{v} , $\mathbf{S}\mathbf{v} = \mathbf{u} \times \mathbf{v}$. Now (21) shows that

$$\mathbf{R}(\mathbf{u}, \theta) = \cos \theta \mathbf{I} + (1 - \cos \theta)\mathbf{u}\mathbf{u}' + \sin \theta \mathbf{S}_{\mathbf{u}}$$

7.6. Shortcuts. We can find the axis of rotation of \mathbf{A} by finding the eigenvector corresponding to the eigenvalue 1. There is a shortcut for finding the axis if the angle of rotation is not π . The matrix $\mathbf{S} = \mathbf{A} - \mathbf{A}'$ is easily seen to be *skew symmetric*, i. e., $\mathbf{S}' = -\mathbf{S}$. So \mathbf{S} must be of the form (22) If the vector $\mathbf{u} = (a, b, c)'$ is not zero, it is parallel to the axis of rotation of \mathbf{A} .

There is also a shortcut for finding the angle θ of rotation. Recall that the trace of a matrix is the sum of the diagonal elements. We have

$$\text{trace}(\mathbf{A}) = 1 + 2 \cos \theta$$

so we can solve for $\cos \theta$ from the trace of \mathbf{A} . See the homework exercise for explanation of why these shortcuts work.