4. Orthogonal Transformations and Rotations

A matrix is defined to be orthogonal if the entries are real and

\[ A' A = I. \]

Condition (1) says that the gram matrix of the sequence of vectors formed by the columns of \( A \) is the identity, so the columns are an orthonormal frame. An orthogonal matrix defines an orthogonal transformation by mutiplying column vectors on the left.

Condition (1) also shows that \( A \) is a rigid motion preserving angles and distances. A matrix satisfying (1) preserves the dot product; the dot product of two vectors is the same before and after an orthogonal transformation. This can be written as

\[ A v \cdot A w = v \cdot w \]

for all vectors \( v \) and \( w \). This is true by the definition (1) of orthogonal matrix since

\[ A v \cdot A w = (A v)' A w = v' A' A w \]
\[ = v' I w = v' w = v \cdot w. \]

Thus lengths and angles are preserved, since they can be written in terms of dot products.

The orthogonal transformation are a group since we can multiply two of them and get an orthogonal transformation. This is because if \( A \) and \( B \) are orthogonal, then \( A' A = I \) and \( B' B = I \). So

\[ (AB)' AB = B' A' A B = I, \]

showing that \( AB \) is also orthogonal. Likewise we can take the inverse of an orthogonal transformation to get an orthogonal transformation.

Orthogonal transformations have determinant 1 or \(-1\) since by (1) and properties of determinant,

\[ (\det A)^2 = \det(A') \det A \]
\[ = \det(A' A) \]
\[ = \det I = 1. \]

4.1. The rotation group. Orthogonal transformations with determinant 1 are called rotations, since they have a fixed axis. This is discussed in more detail below. The rotations also form a group.

If we think of an orthogonal matrix \( A \) as a frame

\[ A = (v_1, v_2, v_3), \]

then the determinant is the scalar triple product

\[ v_1 \cdot (v_2 \times v_3). \]

The frame is right handed if the triple product is 1 and left handed if it is -1. The frame is the image of the right handed standard frame

\[ (e_1, e_2, e_3) = I \]

under the transformation \( A \). Thus \( A \) preserves orientation (right-handedness) if the determinant is 1.
4.1.1. Rotations and cross products. Rotations are orthogonal transformations which preserve orientation. This is equivalent to the fact that they preserve the vector cross product:

\[ \mathbf{A}(\mathbf{v} \times \mathbf{w}) = \mathbf{A}\mathbf{v} \times \mathbf{A}\mathbf{w}, \]

for all rotations \( \mathbf{A} \) and vectors \( \mathbf{v} \) and \( \mathbf{w} \). Recall the right hand rule in the definition of the cross product (fig. 1). The cross product is defined in terms of lengths and angles and right-handedness, all of which remain unchanged after rotation.

![Figure 1](image)

**Figure 1.** The geometric definition of the cross product. The direction of \( \mathbf{v} \times \mathbf{w} \) is determined by the right hand rule. The fingers of the right hand should point from \( \mathbf{v} \) to \( \mathbf{w} \) and the thumb in the direction of the cross-product. The length of the cross product is \( |\mathbf{v}||\mathbf{w}| \sin \theta \).

4.1.2. Two dimensions. In two dimensions, every rotation is of the form

\[ \mathbf{R}(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}. \]

Note that

\[ \mathbf{R}(\theta)\mathbf{R}(\phi) = \mathbf{R}(\theta + \phi) = \mathbf{R}(\phi)\mathbf{R}(\theta), \]

so that rotations in two dimension commute.

4.1.3. Three dimensions. In three dimensions, matrices for rotation about coordinate axes have a form related to the 2 dimensional rotation matrices:

Rotation about the x axis

\[ \mathbf{R}_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix}. \]
Rotation about the $y$ axis
\[
R_y(\theta) = \begin{pmatrix}
\cos \theta & 0 & \sin \theta \\
0 & 1 & 0 \\
-\sin \theta & 0 & \cos \theta
\end{pmatrix}
\]

Rotation about the $z$ axis
\[
R_z(\theta) = \begin{pmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

All rotations are counterclockwise about the axis indicated in the subscript. The rotations $R_x(\theta)$, are exactly the ones that leave the vector $e_3 = (0, 0, 1)'$ fixed and can be identified with rotation in the $xy$ plane. Similarly for $R_y$ and $R_z$.

The rotations $R_x(\theta)$ commute,
\[
R_x(\theta)R_x(\phi) = R_x(\theta + \phi) = R_x(\phi)R_x(\theta).
\]

Similarly rotations $R_y(\theta)$ commute and rotations $R_z(\theta)$ commute. In general, however, rotations in three dimensions do not commute. For example,
\[
R_x(\pi)R_z(\pi/2) = \begin{pmatrix}
0 & -1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & -1
\end{pmatrix}
\]

but
\[
R_z(\pi/2)R_x(\pi) = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & -1
\end{pmatrix}
\]

Since rotations form a group we can get other rotations by multiplying together rotations of the form $R_x$, $R_y$, and $R_z$. It can be shown that any rotation $A$ can be written as a product of three rotations about the $y$ and $z$ axes,
\[
A = R_z(\alpha)R_y(\gamma)R_x(\delta).
\]
The angles $\alpha, \gamma, \delta$ are called Euler angles for the rotation $A$.

The proof that any rotation $A$ can be written as above in terms of Euler angles relies on a simple fact. Any unit vector $u$ can be written in terms of spherical coordinates as
\[
u = \begin{pmatrix}
\sin \phi \cos \theta \\
\sin \phi \sin \theta \\
\cos \phi
\end{pmatrix},
\]
and $u$ can be obtained from $e_3$ by two rotations
\[
u = R_z(\theta)R_y(\phi)e_3.
\]

4.2. Complex form of a rotation. In dimension 2 it is convenient to use complex numbers to write rotations. Rotation by an angle $\theta$ is given by
\[
\zeta \to e^{i\theta} \zeta
\]
where $\zeta = x + iy$. Writing $\bar{\zeta} = x - iy$ the rotation
\[
\begin{pmatrix}
x \\
y
\end{pmatrix} \to \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix} \begin{pmatrix}
x \\
y
\end{pmatrix}
\]
is transformed into
\[
\begin{pmatrix}
\zeta \\
\bar{\zeta}
\end{pmatrix} \rightarrow \begin{pmatrix} e^{i\theta} & 0 \\
0 & e^{-i\theta}
\end{pmatrix} \begin{pmatrix}
\zeta \\
\bar{\zeta}
\end{pmatrix}
\]
which is convenient because the matrix is diagonal. The elements on the diagonal are eigenvalues of
\[
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}.
\]
A similar method works in three dimensions for rotation about the $z$ axis. First perform a change of coordinates. Letting $\zeta = x + iy$ the transformation
\[
\begin{pmatrix} x \\
y \\
z
\end{pmatrix} \rightarrow \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix} x \\
y \\
z
\end{pmatrix}
\]
becomes
\[
\begin{pmatrix}
\zeta \\
\bar{\zeta} \\
z
\end{pmatrix} \rightarrow \begin{pmatrix} e^{i\theta} & 0 & 0 \\
0 & e^{-i\theta} & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
\zeta \\
\bar{\zeta} \\
z
\end{pmatrix}
\]
transforming the matrix for the rotation into a diagonal matrix. The diagonal entries are the eigenvalues of $R_z(\theta)$.

4.3. Eigenvalues of a rotation. First we show that the eigenvalues of an orthogonal matrix have absolute value 1.

To see this, suppose
\begin{equation}
Av = \lambda v
\end{equation}
for a non-zero vector $v$. Taking the adjoint,
\begin{equation}
v^*A^* = \bar{\lambda}v^*
\end{equation}
Since $A$ is real, $A^* = A'$, and multiplying (8) and (9) and using the fact that $A'A = I$ get
\[v^*v = |\lambda|^2 v^*v\]
and hence
\[|\lambda|^2 = 1,
\]
and thus all eigenvalues have absolute value 1.

If $\lambda$ is an eigenvalue of $A$,
\[Av = \lambda v
\]
for some non-zero vector $v$. Taking conjugate of both sides,
\[\bar{A}v = \bar{\lambda}v.
\]
Since $A$ is real, $\bar{A} = A$ so $\bar{\lambda}$ is an eigenvector corresponding to eigenvector $\bar{v}$. It follows that the eigenvalues of an orthogonal matrix $A$ are
\[\pm 1, \quad \lambda, \quad \bar{\lambda},
\]
where $\lambda = e^{i\theta}$. The determinant of $A$ is the product of the eigenvalues, so for a rotation matrix the first eigenvalue above is 1. There is a real eigenvector $u$ corresponding to the eigenvalue 1. This is left as an exercise. The line though this vector $u$ is called the axis of the rotation.

By dividing by the length, we may suppose that $u$ above is a unit vector. As in (6) write $u = Be_3$ where $B$ is a rotation. Since $u$ is left fixed by $A$,
\[ABe_3 = Be_3.
\]
Thus $B^{-1}AB$ leaves $e_3$ fixed and so

$$B^{-1}AB = R_z(\theta)$$

for some angle $\theta$.

We have shown that every rotation is conjugate to a rotation about the $z$ axis. In other words, every rotation $A$ can be written in the form $A = BR_z(\theta)B'$ for a rotation $B$ and some angle $\theta$. This is just a way of saying that by an orientation preserving change of variables we can take the $z$ axis $e_3$ along the axis of the rotation $A$. Since $A$ and $R_z(\theta)$ are conjugate, they have the same eigenvalues, so the eigenvalues of $A$ are $1, e^{i\theta}, e^{-i\theta}$, where $\theta$ is the angle of rotation. The vector $u$ is called the axis of the rotation.

If a rotation $A$ can be written

$$A = BR_z(\theta)B'$$

where $Be_3 = u$ then we write

$$A = R(u, \theta).$$

We can check that $BR_z(\theta)B'$ is the same for any choice of rotation $B$ with $Be_3 = u$, so $R(u, \theta)$ is uniquely defined.

4.4. Properties of rotations. A few of the main properties of rotations are summarized here. In what follows, $u$ is a unit vector and $v$ and $w$ are real vectors, and $A$ is a rotation,

(10) $R(u, \theta) = R(-u, -\theta)$

(11) $Av \cdot Aw = v \cdot w$

(12) $Av \times Aw = A(v \times w)$

(13) $u \cdot (R(u, \theta)v - v) = 0$

(14) $AR(u, \theta)A^{-1} = R(Au, \theta)$

(15) $R(u, \theta)v = \cos \theta(v - (u \cdot v)u) + \sin \theta u \times v + (u \cdot v)u$

We can write (15) in a different form. If $u = (a, b, c)'$ write

$$S_u = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$$

Then

(16) $R(u, \theta) = \cos \theta I + (1 - \cos \theta)uu' + \sin \theta S_u$

The identities (11) and (12) were shown above. The proof (13) and (14) is left as an exercise.

Here is a proof of (15). The idea is to change variables so that $u$ becomes $e_3$ and then the formula becomes obvious. Let $A$ be a rotation such that $u = Ae_3$ and let $w = Av$. Using (14) and applying $A$ to both sides of the equation, it becomes

$$R(e_3, \theta)w = \cos \theta(w - (e_3 \cdot w)e_3) + \sin \theta(e_3 \times w) + (e_3 \cdot w)e_3.$$
Writing \( w = (x, y, z)' \) the equation becomes
\[
R(e_3, \theta)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \cos \theta \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} + \sin \theta \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix},
\]
which is clear from the definition of \( R(e_3, \theta) \).

Here is an example of (16). Let \( u = \frac{1}{\sqrt{3}}(1, 1, 1)' \) and \( \theta = \frac{2\pi}{3} \). Then \( \cos \theta = -\frac{1}{2} \) and \( \sin \theta = \frac{\sqrt{3}}{2} \). Also
\[
S_u = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}, \quad u'u = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.
\]

Then (16) gives
\[
R(u, \theta) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
\]

We see that the rotation permutes the three coordinate axes.