Preliminary Examination in Complex Analysis
January 5, 2004

Solve three from part I and three from part II. Indicate which one you want
graded, or the first ones will be graded.

Part I

1. Let $p$ be a real number not equal to $\pm 1$. Compute using residues
\[ \int_0^{2\pi} \frac{d\theta}{1 - 2p\cos \theta + p^2}. \]

2. Suppose the Bernoulli polynomials are defined by the Taylor expansion
\[ \frac{ze^{wz}}{e^z - 1} = \sum_{k=0}^{\infty} \frac{B_k(w)}{k!} z^k. \]
Find the first three Bernoulli polynomials, $B_0(w)$, $B_1(w)$, $B_2(w)$.

3. Let $f$ be a function with a simple pole at zero with residue $a$. Let $C_R$ be
a segment of a circle given by $z = Re^{i\theta}$ for $\theta_0 \leq \theta \leq \theta_0 + \alpha$. Let
\[ I(R) = \int_{C_R} f(z) \, dz. \]
Show
\[ \lim_{R \to 0} I(R) = \alpha ia. \]

4. Find the residues at all singularities of
\[ \frac{\pi \cot \pi z}{16z^2 - 1}. \]

5. Consider the Zhukovsky’s function
\[ w = f(z) = \frac{1}{2} \left( z + \frac{1}{z} \right). \]
Prove that it is conformal and univalent on the open disk $D$ of radius
1 centered at the origin. Describe the images of the circles $|z| = r$ for
$0 < r < 1$.

Part II

1. Suppose $f$ is a function analytic on some open set containing the closed
disc $\{z : |z| \leq 1\}$. Let $\Gamma$ be the circle $|\zeta| = 1$ and suppose $|z| < 1$. 
a) Show that
\[ 0 = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta) \bar{z}}{1 - \bar{z}\zeta} d\zeta \]

b) Use part a) and Cauchy’s formula to show that
\[ f(z) = \frac{1}{2\pi} \int_{0}^{2\pi} f(e^{it}) \frac{1 - |z|^2}{|1 - \bar{z}e^{it}|^2} dt \]

2. Suppose \( b_n \) is a sequence of complex numbers with \( |b_n| \) increasing to \( \infty \) and
\[ \sum_{n=1}^{\infty} \frac{1}{|b_n|^3} < \infty. \]
Suppose \( R < |b_N| \). Show that the series
\[ \sum_{n=N}^{\infty} \left( \frac{1}{z-b_n} + \frac{1}{b_n} + \frac{z}{b_n^2} \right) \]
converges uniformly in \( |z| < R \) to an analytic function.

3. The order of an elliptic function \( f \) is the multiplicity of the solution of \( f(z) = \infty \) in the fundamental parallelogram \( P \)—in other words, the number of poles, counting multiplicity, inside \( P \). Using known facts about elliptic functions, prove that there cannot exist elliptic functions of order one.

4. Let \( GL(2, \mathbb{C}) \) be the set of \( 2 \times 2 \) matrices with complex entries and non-zero determinant. Let \( \mathcal{M} \) be the set of Möbius (fractional linear) transformations. Consider the map \( \pi \) from \( GL(2, \mathbb{C}) \) to \( \mathcal{M} \) given by
\[ \pi : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto T \]
where
\[ T : z \mapsto \frac{az + b}{cz + d}. \]

a. Show that
\[ \pi(AB) = \pi(A) \circ \pi(B) \]
where \( \circ \) indicates composition of mappings.

b. Find the inverse image of the identity under \( \pi \).