Solve six (and only six!) out of the following list of problems.

1. Let \( f \) be a complex valued function defined on the open region \( D \subset \mathbb{C} \) and write \( f = u + iv \), where \( u \) and \( v \) are the real and imaginary parts, respectively. Prove the following fact:

\[ f \text{ is analytic at } z \in D \text{ if } u \text{ and } v \text{ are differentiable at } z = x + iy \]

and the Cauchy-Riemann equations \( u_x = v_y \) and \( u_y = -v_x \) are satisfied at \( z \).

2. A pair of real valued harmonic functions \( u \) and \( v \) defined on a region \( D \subset \mathbb{C} \) are said to be conjugate if they are the real and imaginary parts of an analytic function \( f = u + iv \).

   a. Prove that if \( u \) is harmonic on a simply connected region \( D \) then there exists a harmonic conjugate.

   b. Give a counterexample to explain why the simple connectedness hypothesis is required (Be sure to explain what “simply connected” means.)

3. Consider the following form of Jordan’s lemma:

Let the function \( f(z) \) be analytic in the upper half plane \( \mathbb{H} = \{ z : \text{Im}(z) > 0 \} \) with the possible exception of a finite number of isolated singular points, and let it tend to zero as \( |z| \to \infty \), uniformly in \( \arg z \in [0, \pi] \). Then for \( a > 0 \)

\[
\lim_{R \to \infty} \int_{C_R} e^{iaz} f(z) \, dz = 0,
\]

where \( C_R \) is the semicircular arc \( \{ z : |z| = R, \text{Im}(z) > 0 \} \) in \( \mathbb{H} \).

   a. Use Jordan’s lemma and the residue technique to compute the improper integral

\[
I = \int_0^\infty \frac{\cos x}{x^2 + 4} \, dx.
\]

   Be sure to explain why and how you apply Jordan’s lemma and justify all your main steps.

   b. Prove Jordan’s lemma. (Hint: Use the inequality \( \sin \theta \geq \frac{2}{\pi} \theta \) for \( 0 \leq \theta \leq \frac{\pi}{2} \).)

4. Let \( f(z) = \frac{(z-1)^2}{z^2+1} \cdot \exp\left(\frac{1}{z-1}\right) \) and \( g(z) = \pi^2 z^2 \csc^2(\pi z) \).

   a. Determine and classify all the isolated singularities of \( f \) and \( g \) on the Riemann Sphere \( \mathbb{P}^1 = \mathbb{C} \cup \{ \infty \} \). (You will have to explain how to define “analytic” at \( \infty \).)
b. Determine the singular parts and residues of $f$ and $g$ at those points.

5. Determine the number of zeros of the polynomial $P(z) = z^{87} + z^{36} - 4z^5 + 1$ contained in the open unit disk $\mathbb{D} = \{ z : |z| < 1 \}$.

6. Find explicitly a conformal map from the open unit disk $\mathbb{D}$ to the half plane $H$ containing $i$ and bounded by the line $L$ passing through $-1 - i$, the origin and $1 + i$.

7. Consider the Zhukovsky’s function
   \[ w = f(z) = \frac{1}{2} \left( z + \frac{1}{z} \right). \]
   Prove that it is conformal and univalent on the open disk $\mathbb{D}$ of radius 1 centered at the origin. Determine the image $\Delta = f(\mathbb{D})$ and discuss the behavior of $f$ at the boundary.

8. a. Fix a real number $R > 0$. Show that for $n \in \mathbb{N}$, $n > R/2$, and $z \in D_R = \{ z : |z| < R \}$,
   \[ \left| \frac{1}{z^2 + 4n^2} \right| \leq \frac{1}{4n^2 \left| 1 - \frac{R^2}{R^2} \right|}, \]
   where $R'$ is any real number such that $n > R'/2 > R/2$.
   b. Prove that
   \[ f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{1}{z^2 + 4n^2} \]
   defines a meromorphic function in the complex plane. What are the poles and corresponding residues?

9. The order of an elliptic function $f$ is the multiplicity of the solution of $f(z) = \infty$ in the fundamental parallelogram $P$—in other words, the number of poles, counting multiplicity, inside $P$. Using known facts about elliptic functions, prove that there cannot exist elliptic functions of order one.

10. Find the genus and branch points for the covering of $\mathbb{P}^1$ by the Riemann surface associated to the Fermat curve $C$ defined by the equation $z^n + w^n = 1$, where the covering map $\pi: C \to \mathbb{P}^1$ is $(z, w) \mapsto z$. (Your results must be expressed in terms of the integer $n$.)