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The Binomial Theorem

The Binomial Theorem provides a method for the expansion of a binomial raised to a power. For this class, we will be looking at binomials raised to whole number powers, in the form $(A+B)^n$.

The Binomial Theorem

$$(A+B)^n = \sum_{r=0}^{n} \binom{n}{r} A^{n-r} B^r$$

This is a pretty intimidating looking formula, but it actually represents a straightforward process for expanding these binomials-to-powers. Let's take it apart and examine each of the pieces.

- First of all we see Σ. We recognize this as summation notation—that there will be a sum of terms in this expansion. We start our index of summation at 0 and stop at n, which will give us $n+1$ terms in the sum.
- Next we see $\binom{n}{r}$. This is a symbol that represents a numerical coefficient for each term. We will see below how to compute each number from this symbol.
- Now we come to A^{n-r}. This means each term will contain a power of A, with the first term containing A^n, and the exponent on A decreasing by 1 in each subsequent term.
- B^r tells us each term will contain B raised to a power, starting at B^0 in the first term, with the exponent increasing by 1 in each subsequent term.

Combinations

The symbol $\binom{n}{r}$ is used often in Statistics. It represents the number of ways a sample of size r can be selected from a population of size n. In statistical research, one may need to know the number of ways a sample of size 10 can be selected from a population of 100, for instance. The symbol $\binom{n}{r}$ represents the number of possible *combinations* of n objects selected in groups of size r. The formula is

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Factorials

The formula above introduces yet another symbol, $n!$, which is read as "n-factorial." The factorial of a positive integer n is the product of all the integers from 1 to n, usually written in descending order:

$$n! = n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$$

For example,

$$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$$

By definition, $0! = 1$. Intuitively, it might not seem logical that $0! = 1$, but it is (we won't justify the fact here), and it helps the combinations formula give a proper result in certain circumstances.

$$\binom{5}{5} = \frac{5!}{5!(5-5)!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \ \cdot \ 0!} = 1$$

This represents the number of ways that a sample of size 5 can be selected from a population of size 5. There is only one way to do that—sample the entire population.

Your calculator might have a built-in function for computing factorials. Your calculator might also have a built-in function for the combinations formula, as well.

Due date: Thu Sep 15 11:59:59 pm 2016 (EDT)

Evaluate the factorial.
$7!$

Tries 0/5

Evaluate

$$\binom{23}{9}$$

(Combinations of 23 objects selected 9 at a time.)

Tries 0/5

Binomial Theorem Example

Expand $(x+y)^5$ using the binomial theorem.

$$(x+y)^5 = \binom{5}{0}x^5y^0 + \binom{5}{1}x^4y^1\binom{5}{2}x^3y^2 + \binom{5}{3}x^2y^3 + \binom{5}{4}x^1y^4 + \binom{5}{5}x^0y^5$$

Let's crunch the numbers

$$\binom{5}{0} = \frac{5!}{0!(5-0)!} = \frac{5\cdot4\cdot3\cdot2\cdot1}{0!(5\cdot4\cdot3\cdot2\cdot1)} = 1$$

$$\binom{5}{1} = \frac{5!}{1!(5-1)!} = \frac{5\cdot4\cdot3\cdot2\cdot1}{1(4\cdot3\cdot2\cdot1)} = 5$$

$$\binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5\cdot4\cdot3\cdot2\cdot1}{(2\cdot1)(3\cdot2\cdot1)} = 10$$

$$\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5\cdot4\cdot3\cdot2\cdot1}{(3\cdot2\cdot1)(2\cdot1)} = 10$$

$$\binom{5}{4} = \frac{5!}{4!(5-4)!} = \frac{5\cdot4\cdot3\cdot2\cdot1}{(4\cdot3\cdot2\cdot1)1} = 5$$

$$\binom{5}{5} = \frac{5!}{5!(5-5)!} = \frac{5\cdot4\cdot3\cdot2\cdot1}{(5\cdot4\cdot3\cdot2\cdot1)0!} = 1$$

Now put it all together, using the fact that $x^0 = 1$ and $y^0 = 1$:

$$(x+y)^5 = 1x^5y^0 + 5x^4y^1 + 10x^3y^2 + 10x^2y^3 + 5x^1y^4 + 1x^0y^5$$

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

Another Example

Expand $(2x-y)^6$ using the binomial theorem.

$$\begin{aligned}(2x-y)^6 = &\binom{6}{0}(2x)^6(-y)^0 + \binom{6}{1}(2x)^5(-y)^1 + \binom{6}{2}(2x)^4(-y)^2 + \binom{6}{3}(2x)^3(-y)^3\\ &+\binom{6}{4}(2x)^2(-y)^4 + \binom{6}{5}(2x)^1(-y)^5 + \binom{6}{6}(2x)^0(-y)^6\end{aligned}$$

$$\binom{6}{0} = \frac{6!}{0!(6-0)!} = \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{0!(6\cdot5\cdot4\cdot3\cdot2\cdot1)} = 1$$

$$\binom{6}{1} = \frac{6!}{1!(6-1)!} = \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{1(5\cdot4\cdot3\cdot2\cdot1)} = 6$$

$$\binom{6}{2} = \frac{6!}{2!(6-2)!} = \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{(2\cdot1)(4\cdot3\cdot2\cdot1)} = 15$$

$$\binom{6}{3} = \frac{6!}{3!(6-3)!} = \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{(3\cdot2\cdot1)(3\cdot2\cdot1)} = 20$$

$$\binom{6}{4} = \frac{6!}{4!(6-4)!} = \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{(4\cdot3\cdot2\cdot1)(2\cdot1)} = 15$$

$$\binom{6}{5} = \frac{6!}{5!(6-5)!} = \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{(6\cdot5\cdot4\cdot3\cdot2\cdot1)1} = 6$$

$$\binom{6}{6} = \frac{6!}{6!(6-6)!} = \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{(6\cdot5\cdot4\cdot3\cdot2\cdot1)0!} = 1$$

$$\begin{aligned}(2x-y)^6 &= 1(2x)^6(-y)^0 + 6(2x)^5(-y)^1 + 15(2x)^4(-y)^2 + 20(2x)^3(-y)^3 + 15(2x)^2(-y)^4 + 6(2x)^1(-y)^5 + 1(2x)^0(-y)^6\\ &= 64x^6 + 6(32x^5)(-y)^1 + 15(16x^4)(-y)^2 + 20(8x^3)(-y)^3 + 15(4x^2)(-y)^4 + 6(2x)(-y)^5 + (-y)^6\\ &= 64x^6 - 192x^5y + 240x^4y^2 - 160x^3y^3 + 60x^2y^4 - 12xy^5 + y^6\end{aligned}$$

Pascal's Triangle

Pascal's Triangle is a device that we can use to compute the binomial coefficients of $(A+B)^n$ for small n. It is named for the French mathematician Blaise Pascal. Pascal lived in the 17$^{\text{th}}$ century, but Pascal's Triangle was already known to the Chinese in the 11$^{\text{th}}$ century.

Pascal's Triangle is a triangular arrangement of numbers. The first two rows are an arrangement of ones:

$$\begin{array}{ccc} & 1 & \\ 1 & & 1 \end{array}$$

New rows are added to the bottom of the triangle, and each row begins and ends with 1. Each number in the interior of a row is the sum of the two numbers above it.

$$\begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ 1 & & 2 & & 1 \end{array}$$

Repeating the process, it is very fast to write out the first several rows of the triangle.

$$\begin{array}{ccccccccccccccc} & & & & & & & 1 & & & & & & & \\ & & & & & & 1 & & 1 & & & & & & \\ & & & & & 1 & & 2 & & 1 & & & & & \\ & & & & 1 & & 3 & & 3 & & 1 & & & & \\ & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\ & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\ & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\ 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 \end{array}$$

We can continue adding rows to Pascal's Triangle indefinitely. What's remarkable is that this arrangement computes for us the binomial coefficients. Each row of Pascal's Triangle corresponds to an exponent n:

$$\begin{array}{cccccccccccccccc} & & & & & & & 1 & & & & & & & & n{=}0 \\ & & & & & & 1 & & 1 & & & & & & & n{=}1 \\ & & & & & 1 & & 2 & & 1 & & & & & & n{=}2 \\ & & & & 1 & & 3 & & 3 & & 1 & & & & & n{=}3 \\ & & & 1 & & 4 & & 6 & & 4 & & 1 & & & & n{=}4 \\ & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & n{=}5 \\ & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & n{=}6 \\ 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 & n{=}7 \end{array}$$

Example

Use Pascal's Triangle and the Binomial Theorem to expand $(x+2)^4$.

Using the row of Pascal's Triangle corresponding to $n=4$,

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

$$\begin{aligned} (x+2)^4 &= 1 \cdot x^4 \cdot 2^0 + 4 \cdot x^3 \cdot 2^1 + 6 \cdot x^2 \cdot 2^2 + 4 \cdot x^1 \cdot 2^3 + 1 \cdot x^0 \cdot 2^4 \\ &= x^4 + 8x^3 + 24x^2 + 32x + 16 \end{aligned}$$

Finding a Particular Term in a Binomial Expansion

Problem

Find the term containing x^8 in the expansion of $(x-2y)^{20}$.

Since we are looking for only one term in the expansion, we do not need to do the complete expansion. Furthermore, we don't want to write out enough rows of Pascal's Triangle to get the binomial coefficients for $n = 20$. However, using the format of the Binomial Theorem, it is easy to set up the form of the term containing x^8. Let's begin with the structure of the term.

$$\binom{20}{?} x^{(?)}(-2y)^{(?)}$$

We know we want the term containing x^8, so we can fill in that exponent. The two exponents must sum to 20, so we know the exponent on $(-2y)$ must be 12. Then the bottom number in the binomial coefficient can be either of the two exponents.

$$\binom{20}{12} x^8(-2y)^{12}$$

$$125,970x^8(4096y^{12})$$

$$515,973,120x^8y^{12}$$

Another Problem

Find the fifteenth term in the expansion of $(2x-y)^{18}$.

Using the Binomial Theorem, set up the form of the term we want.

$$\binom{18}{?}(2x)^{(?)}(-y)^{(?)}$$

In the first term of the expansion, the exponent on $(-y)$ is 0, and the exponent increases by 1 in each subsequent term. Therefore, the exponent on $(-y)$ in the fifteenth term must be 14. Filling in the rest of the parts, we get

$$\binom{18}{14}(2x)^4(-y)^{14}$$

$$3060(16x^4)(y^{14})$$

$$48,960x^4y^{14}$$