1.2 #5 If $\mathcal{M}(\mathcal{E})$ is the σ-algebra generated by $\mathcal{E}$, then $\mathcal{M}(\mathcal{E})$ is the union of the σ-algebras generated by $\mathcal{F}$ as $\mathcal{F}$ ranges over all countable subsets of $\mathcal{E}$. (Hint: Show that the latter object is a σ-algebra).

Proof. Let

$$\mathcal{M} = \bigcup_{\substack{\mathcal{F}\subset\mathcal{E} \\ \mathcal{F} \text{ is countable}}} \mathcal{M}(\mathcal{F})$$

so that our task is to show that $\mathcal{M} = \mathcal{M}(\mathcal{E})$.

First we show that $\mathcal{M} \subset \mathcal{M}(\mathcal{E})$. It suffices to observe that for every countable subset $\mathcal{F}$ of $\mathcal{E}$ that $\mathcal{M}(\mathcal{F}) \subset \mathcal{M}(\mathcal{E})$. But, since $\mathcal{F} \subset \mathcal{E} \subset \mathcal{M}(\mathcal{E})$ we see that $\mathcal{M}(\mathcal{E})$ is a σ-algebra containing $\mathcal{F}$ and so $\mathcal{M}(\mathcal{F})$ (the intersection of all σ-algebras containing $\mathcal{F}$) must be contained in $\mathcal{M}(\mathcal{E})$.

To see that $\mathcal{M} \subset \mathcal{E}$ we first note that it suffices to show that $\mathcal{M}$ is a σ-algebra. Indeed, then we would have (since for each $E \in \mathcal{E}$, $\{E\}$ is countable and so $E \in \mathcal{M}(\{E\}) \subset \mathcal{M}$) that $\mathcal{M}$ is a σ-algebra containing $\mathcal{E}$ and so, by the logic of the previous paragraph we would thus have $\mathcal{M}(\mathcal{E}) \subset \mathcal{M}$.

Finally, we check that $\mathcal{M}$ is a σ-algebra. First, suppose that $E \in \mathcal{M}$. Then, $E \in \mathcal{M}(\mathcal{F})$ for some countable $\mathcal{F} \subset \mathcal{E}$. Since $\mathcal{M}(\mathcal{F})$ is a σ-algebra, we have $E^c \in \mathcal{M}(\mathcal{F}) \subset \mathcal{M}$ as desired. Now, suppose that $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M}$. Then, for every j we have $E_j \in \mathcal{M}(\mathcal{F}_j)$ for some countable subset $\mathcal{F}_j$ of $\mathcal{E}$. Setting

$$\mathcal{F} = \bigcup_{j=1}^{\infty} \mathcal{F}_j$$

we have that $\mathcal{F}$ is countable (a countable union of countable sets is countable!) and each $E_j \in \mathcal{M}(\mathcal{F}_j) \subset \mathcal{M}(\mathcal{F})$ (again, the logic of the second paragraph shows that $\mathcal{F}_j \subset \mathcal{F} \Rightarrow \mathcal{M}(\mathcal{F}_j) \subset \mathcal{M}(\mathcal{F})$). Thus, since $\mathcal{M}(\mathcal{F})$ is a σ-algebra we have $\bigcup_{j=1}^{\infty} E_j \in \mathcal{M}(\mathcal{F}) \subset \mathcal{M}$ as desired. $\square$