1.3 #11 A finitely additive measure μ is a measure if and only if it is continuous from below as in Theorem 1.8c. If $\mu(X) < \infty$, then μ is a measure if and only if it is continuous from above as in Theorem 1.8d.

Proof. Let μ be a finitely additive measure defined on a σ-algebra $\mathcal{M}$.

Beginning with the first statement, it suffices to show, by Theorem 1.8c, that if μ is continuous from below then it is a measure. For this, we need to demonstrate that it is countably additive. Suppose that $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M}$ is a collection of pairwise disjoint sets. Then, since $\bigcup_{j=1}^{n} E_j \subset \bigcup_{j=1}^{n+1} E_j$ for each n, continuity from below gives

$$\mu(\bigcup_{j=1}^{\infty} E_j) = \lim_{n\to\infty} \mu(\bigcup_{j=1}^{n} E_j).$$

On the other hand, since μ is finitely additive we have

$$\mu(\bigcup_{j=1}^{n} E_j) = \sum_{j=1}^{n} \mu(E_j)$$

and so, since

$$\lim_{n\to\infty} \sum_{j=1}^{n} \mu(E_j) = \sum_{j=1}^{\infty} \mu(E_j),$$

we are done.

For the second statement, assume instead that μ is continuous from above and $\mu(X) < \infty$. Then, using the fact that μ is finitely additive and $\mu(X) < \infty$

$$\mu(\bigcup_{j=1}^{\infty} E_j) = \mu(X) - \mu(\left(\bigcup_{j=1}^{\infty} E_j\right)^c).$$

Then

$$\mu(\left(\bigcup_{j=1}^{\infty} E_j\right)^c) = \mu(\bigcap_{j=1}^{\infty} E_j^c)$$

and by continuity from above

$$\mu(\bigcap_{j=1}^{\infty} E_j^c) = \lim_{n\to\infty} \mu(\bigcap_{j=1}^{n} E_j^c)$$

but by finite additivity

$$\mu(\bigcap_{j=1}^{n} E_j^c) = \mu(\left(\bigcup_{j=1}^{n} E_j\right)^c) = \mu(X) - \mu(\bigcup_{j=1}^{n} E_j) = \mu(X) - \sum_{j=1}^{n} \mu(E_j)$$

and taking limits we are done. $\square$