1.3 #11 Let $\mathcal{A} \subset 2^X$ be an algebra, $\mathcal{A}_\sigma$ the collection of countable unions of sets in $\mathcal{A}$, and $\mathcal{A}_{\sigma\delta}$ the collection of countable intersections of sets in $\mathcal{A}_\sigma$. Let μ_0 be a premeasure on $\mathcal{A}$ and μ^* the induced outer measure.

(a) For any $E \subset X$ and $\epsilon > 0$ there exists $A \in A_\sigma$ with $E \subset A$ and $\mu^*(A) \leq \mu^*(E) + \epsilon$

(b) If $\mu^*(E) < \infty$, then E is μ^*-measurable if and only if there exists $B \in A_{\sigma\delta}$ with $E \subset B$ and $\mu^*(B \setminus E) = 0$

(c) If μ_0 is σ-finite, then the restriction $\mu^*(E) < \infty$ in (b) is superfluous.

Proof. For (a), let $E \subset X$ and $\epsilon > 0$. Since the infimum in the definition of $\mu^*(E)$ is the *greatest* lower bound for

$$\left\{ \sum_{j=1}^{\infty} \mu_0(A_j) : \{A_j\}_{j=1}^{\infty} \subset \mathcal{A}, \text{ and } E \subset \bigcup_{j=1}^{\infty} A_j \right\} \tag{1}$$

we have that $\mu^*(E) + \epsilon$ is not a lower bound, and so there is a sequence $\{A_j\}_{j=1}^{\infty} \subset \mathcal{A}$ with $E \subset \bigcup_{j=1}^{\infty} A_j$ and $\sum_{j=1}^{\infty} \mu_0(A_j) < \mu^*(E) + \epsilon$. By Proposition 1.13, we have $\mu^*(A_j) = \mu_0(A_j)$ for each j, and so by countable subadditivity $\mu^*(\bigcup_{j=1}^{\infty} A_j) \leq \mu^*(E) + \epsilon$ as desired.

For the"if" direction of (b), first suppose that $E \subset X$ (allowing the possibility that $\mu^*(E) = \infty$) is such that there is a $B \in A_{\sigma\delta}$ with $E \subset B$ and $\mu^*(B \setminus E) = 0$. Letting $\mathcal{M}$ denote the σ-algebra of μ^*-measurable sets, we have $\mathcal{A} \subset \mathcal{M}$ by Proposition 1.13. Thus (since $\mathcal{M}$ is closed under countable unions and intersections) $\mathcal{A}_{\sigma\delta} \subset M$ and so $B \in \mathcal{M}$. Next observe (as in the end of the proof of Carathéodory's theorem) that for every $F \subset X$

$$\mu^*(F \cap (B \setminus E)) + \mu^*(F \cap (B \setminus E)^c) \leq \mu^*(B \setminus E) + \mu^*(F) = \mu^*(F)$$

and so $B \setminus E \in \mathcal{M}$ (above we used the monotonicity of μ^* twice). Finally, since $E = B \setminus (B \setminus E)$ and the latter two sets are in $\mathcal{M}$, we also have that $E \in \mathcal{M}$.

For the"only if" direction of (b), suppose that $E \subset X$ is μ^*-measurable with $\mu^*(E) < \infty$. By part (a), we can find, for each $n \in \mathbb{N}$, a $B_n \in \mathcal{A}_\sigma$ with $E \subset B_n$ and $\mu^*(B_n) \leq \mu^*(E) + \frac{1}{n}$. Letting $B = \bigcap_{j=1}^{\infty} B_n$, we have $E \subset B$, $B \in \mathcal{A}_{\sigma\delta}$, and (by two applications of monotonicity) $\mu^*(B) = \mu^*(E)$. Since E is μ^*-measurable

$$\mu^*(E) = \mu^*(B) = \mu^*(B \setminus E) + \mu^*(B \cap E) = \mu^*(B \setminus E) + \mu^*(E)$$

and since $\mu^*(E) < \infty$ we may subtract to see that $\mu^*(B \setminus E) = 0$.

Finally, we consider the "only if" direction of (b) with an assumption of σ-finiteness for μ_0 replacing the requirement that $\mu^*(E) < \infty$. By σ-finiteness, we may choose $\{D_l\}_{l=1}^{\infty} \subset \mathcal{A}$ with $X \subset \bigcup_{l=1}^{\infty} D_l$ and $\mu_0(D_l) = \mu^*(D_l) < \infty$ for each l. After applying the"disjointification trick", we may assume that the D_l's are pairwise disjoint. For each l, n, apply part

(a) to find $B_{l,n} \in \mathcal{A}_\sigma$ with $(E \cap D_l) \subset B_{l,n}$ and $\mu^*(B_{l,n}) \leq \mu^*(E \cap D_l) + \frac{1}{n}$. After possibly intersecting $B_{l,n}$ with D_l, we may further assume that $B_{l,n} \subset D_l$ (note that after this intersection, $B_{l,n}$ is still an $\mathcal{A}_\sigma$ set). Let $B = \bigcap_{n=1}^\infty \bigcup_{l=1}^\infty B_{l,n}$ so that $E \subset B \in \mathcal{A}_{\sigma\delta}$. For each l,

$$B \cap D_l = \cap_{n=1}^\infty B_{l,n}$$

and, as in the previous paragraph, $\mu^*(B \cap D_l) = \mu^*(E \cap D_l)$. Since $\mu^*(E \cap D_l) < \infty$, we have $\mu^*((B \cap D_l) \setminus (E \cap D_l)) = 0$. Then

$$\mu^*(B \setminus E) = \mu^*(\bigcup_{l=1}^\infty (B \cap D_l) \setminus (E \cap D_l)) = \sum_{l=1}^\infty \mu^*((B \cap D_l) \setminus (E \cap D_l)) = 0$$

$\square$