2.1 #10 The following implications are valid if and only if the measure μ is complete

(a) If f is measurable and $f = g$ μ-a.e. then g is measurable

(b) If f_n is measurable for $n \in \mathbb{N}$ and $f_n \to f$ μ-a.e. then f is measurable

Proof.

if:

(a) We need to show that for every Borel set $B \subset \mathbb{R}$ we have $g^{-1}(B)$ measurable. Since $f = g$ a.e. there is a measurable set E with $\mu(E) = 0$ such that for all $x \notin E$ we have $f(x) = g(x)$. Then

$$g^{-1}(B) = (f^{-1}(B) \setminus E) \cup (E \cap g^{-1}(B)).$$

Since f is measurable, we have $f^{-1}(B)$ measurable, and since μ is complete, we have $E \cap g^{-1}(B)$ measurable. Thus $g^{-1}(B)$ is measurable.

(b) Let $\tilde{f} = \limsup_{n\to\infty} f_n$. Then since each f_n is measurable, $\tilde{f}$ is measurable. Since $f_n \to f$ a.e., we have $\tilde{f} = f$ a.e. and so, by (a) f is measurable.

only if:

Suppose that μ is not complete. Then there is a measurable set E with $\mu(E) = 0$ and a set $F \subset E$ such that F is not measurable. Then, for (a) note that 1_F is not measurable and $1_F = 0$ a.e. Similarly, for (b) define $f_n = 0$ for all n and $f = 1_F$.

□