2.2 #15 If $\{f_n\}_{n=1}^{\infty} \subset L^+$, f_n decreases pointwise to f, and $\int f_1 \, d\mu < \infty$, then

$$\int f \, d\mu = \lim_{n\to\infty} \int f \, d\mu. \tag{1}$$

Proof. Note that $\{f_1 - f_n\}_{n=1}^{\infty} \subset L^+$ is increasing and so

$$\begin{aligned} \int f_1 - \int f = \int f_1 - f = \int \lim_{n\to\infty} f_1 - f_n = \lim_{n\to\infty} \int f_1 - f_n \\ = \lim_{n\to\infty} \int f_1 - \int f_n = \int f_1 - \lim_{n\to\infty} \int f_n \end{aligned}$$

where the third identity follows from the monotone convergence theorem. Since $\int f_1 < \infty$, we can cancel it from both sides and obtain (1) as desired. □