2.4 #44 (Lusin's Theorem) If $f : [a,b] \to \mathbb{C}$ is Lebesgue measurable and $\epsilon > 0$, there is a compact set $E \subset [a,b]$ such that $\mu(E^c) < \epsilon$ and $f\big|_E$ is continuous (Hint: Use Egoroff's theorem and Theorem 2.26)

Proof. First, note that

$$[a,b] = \bigcup_{n\in\mathbb{N}} \{|f| \le n\}$$

and so, by continuity from below, we can find an n_0 such that letting $E_1 = \{|f| \le n_0\}$ we have $m(E_1^c) < \epsilon/3$. Setting $\tilde{f}$ equal to f on E_1 and 0 on E_1^c we then have $\tilde{f}$ integrable on $[a,b]$ (since it is measurable, bounded, and $[a,b]$ has finite measure).

Since $\tilde{f}$ is integrable, Theorem 2.26 gives a sequence $\{\phi_n\}_{n=1}^\infty$ of continuous functions on $[a,b]$ such that $\phi_n \to \tilde{f}$ in L^1. Then by Theorem 2.30 there is a subsequence $\{\phi_{n_j}\}_{j=1}^\infty$ such that $\phi_{n_j} \to \tilde{f}$ pointwise a.e.. By Egoroff's theorem there is a set $E_2 \subset E_1$ with $m(E_2^c) < \frac{2\epsilon}{3}$ such that $\phi_{n_j} \to \tilde{f}$ uniformly on E_2. Then by Theorem 1.18 there is a set compact set $E_3 \subset E_2$ with $m(E_3^c) < \epsilon$. Since $f = \tilde{f}$ on E_3, it is the uniform limit of continuous functions and hence continuous on E_3. □

2.3 #38b Suppose $f_n \to f$, $g_n \to g$ in measure and $\mu(X) < \infty$. Show that $f_n g_n \to fg$ in measure.

Proof. Let $\epsilon, \eta > 0$. We need to find an N such that for all $n \ge N$

$$\mu(\{|f_n g_n - fg| \ge \epsilon\}) \le \eta.$$

Since

$$X = \bigcup_{M\in\mathbb{N}} \{|f| \le M\}$$

and $\mu(X) < \infty$ we can find an M_1 so that, letting $E_1 = \{|f| \le M_1\}$, we have $\mu(E_1) \le \eta/10$. We define E_2 and M_2 analogously, but with g in place of f.

Since $g_n \to g$ in measure, we can find an N_1 so that for $n \ge N_1$, $\mu(\{|g_n - g| \ge \epsilon/(2M_1)\}) \le \eta/10$ and an N_2 so that for $n \ge N_2$, $\mu(\{|g_n - g| \ge M_2\}) \le \eta/10$. Since $f_n \to f$ in measure, we can find an N_3 so that for $n \ge N_3$, $\mu(\{|f_n - f| \ge \epsilon/(4M_2)\})$.

Note that

$$|f_n g_n - fg| \le |f| \cdot |g_n - g| + |g_n| \cdot |f_n - f|$$

and that if $x \notin E_2$ and $|g_n(x) - g(x)| \le M_2$ then $|g_n| \le 2M_2$. So

$$\{|f_n g_n - fg| \ge \epsilon\} \subset F_n$$

where

$$F_n = E_1 \cup E_2 \cup \{|g_n - g| \ge \epsilon/(2M_1)\} \cup \{|g_n - g| \ge M_2\} \cup \{|f_n - f| \ge \epsilon/(4M_2)\}$$

But, if $n \ge \max(N_1, N_2, N_3)$ then $\mu(F_n) \le 5\eta/10$ as desired. □