2.5 #49 (The Fubini-Tonelli theorem for Complete Measures) Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be complete, σ-finite measure spaces, and let $(X \times Y, \mathcal{L}, \lambda)$ be the completion of $(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \times \nu)$. If f is $\mathcal{L}$-measurable and either (a) $f \geq 0$ or (b) $f \in L^1(\lambda)$, then f_x is $\mathcal{N}$-measurable for a.e. x and f^y is $\mathcal{M}$-measurable for a.e. y and in case (b) f_x and f^y are also integrable for a.e. x and y. Moreover $x \to \int f_x d\nu$ and $y \to \int f^y d\mu$ are measurable, and in case (b) also integrable and

$$\int f d\lambda = \int \int f(x, y)\ d\mu(x) d\nu(y) = \int \int f(x, y)\ d\nu(y) d\mu(x).$$

Proof. By Proposition 2.12 in the book, there is an $\mathcal{M} \otimes \mathcal{N}$-measurable function g such that $g = f, \lambda$ a.e.. Let $h = f - g$ so that h is λ-measurable and $h = 0, \lambda$ a.e. By the standard Fubini-Tonelli theorem (applied to g) it suffices to prove all claims with h in place of f.

Since $h = 0, \lambda$ a.e. and λ is the completion of $\mu \times \nu$, there is an $E \in \mathcal{M} \otimes \mathcal{N}$ with $\mu \times \nu(E) = 0$ such that

$$\{f \neq 0\} \subset E.$$

Assuming for the moment:

Claim 0.1. *If $E \in \mathcal{M} \otimes \mathcal{N}$ and $\mu \times \nu(E) = 0$, then $\nu(E_x) = \mu(E^y) = 0$ for a.e. x and y.*

it follows that for μ a.e. x we have $h_x = 0$ ν a.e. and that for ν a.e. y we have $h^y = 0$ μ a.e. Since ν and μ are complete, this implies that h_x and h^y are measurable for μ a.e. x and ν a.e. y. Assuming for the moment

Claim 0.2. *If h is $\mathcal{L}$-measurable and $h = 0$ λ a.e. then h_x and h^y are integrable for a.e. x and y and $\int h_x\ d\nu = \int h_x = 0$ for a.e. x and y*

the proof is evidently finished since the a.e. defined functions

$$x \to \int h_x\ d\nu \text{ and } y \to \int h^y d\mu$$

are 0 a.e. and hence measurable (using again the completeness of μ and ν) and integrable with integrals 0 (matching the integral of h w.r.t. λ).

The first claim is immediate from Tonelli's theorem applied to the characteristic function of E, since a nonnegative function with integral 0 must be 0 a.e.

The second claim is immediate from the paragraph between the first and second claims (applied in some parts to $|h|$ instead of h) and the fact that the integrable of a measurable function which is 0 a.e. is 0. $\square$