Proposition 0.1. *Suppose that $E \in \mathcal{L}$ and $s \in \mathbb{R}$. Then $E + s \in \mathcal{L}$ and*

$$m(E + s) = m(E) \tag{1}$$

Proof. We first establish some notation for sums. Recall that if $E \subset \mathbb{R}$ and $s \in \mathbb{R}$ we write $E + s := \{x + s : x \in E\}$. Then, if $\mathcal{M}$ is a set of sets we may write $\mathcal{M} + x := \{E + x : E \in \mathcal{M}\}$ and if $\mathfrak{M}$ is a set of sets of sets we may write $\mathfrak{M} + x := \{\mathcal{M} + x : \mathcal{M} \in \mathfrak{M}\}$.

We begin by observing that the Borel sets are translation invariant, i.e.

$$\mathcal{B}_{\mathbb{R}} + x = \mathcal{B}.$$

Let $\mathfrak{M}$ denote the set of all σ-algebras on $\mathbb{R}$ which contain the collection $\mathcal{O}$ of open subsets of $\mathbb{R}$, so that

$$\mathcal{B}_{\mathbb{R}} = \bigcap_{\mathcal{M} \in \mathfrak{M}} \mathcal{M}.$$

Then

$$\mathcal{B}_{\mathbb{R}} + x = \bigcap_{\mathcal{M} \in \mathfrak{M}} (\mathcal{M} + x) = \bigcap_{\mathcal{M} \in (\mathfrak{M} + x)} \mathcal{M}$$

so it will suffice to show that $\mathfrak{M} + x = \mathfrak{M}$. To see this, first note that for each $\mathcal{M} \in \mathfrak{M} + x$, $\mathcal{M}$ is a σ-algebra since the complement of the translate is the translate of the complement and the union of the translates is the translate of the union. Next, note that each $\mathcal{M} \in \mathfrak{M} + x$ contains $\mathcal{O} + x = \mathcal{O}$. Thus, $\mathfrak{M} + x \subset \mathfrak{M}$. By the same reasoning $\mathfrak{M} = (\mathfrak{M} + x) - x \subset \mathfrak{M} + x$ and so $\mathfrak{M} + x = \mathfrak{M}$.

Letting

$$m_s(E) := m(E + s)$$

we have, from the previous paragraph, that the domain of m_s contains $\mathcal{B}_{\mathbb{R}}$. Since "disjointness is invariant under translations", it is easy to check that $m_s(E)$ is in fact a measure on $\mathcal{B}_{\mathbb{R}}$. Given $a, b \in \mathbb{R}$ we have

$$\begin{aligned} m_s((a, b]) = m((a, b] + s) = m((a + s, b + s]) &= (b + s) - (a + s) \\ &= b - a = m((a, b]) \end{aligned}$$

where the last and third to last identities follow from Proposition 1.13 in the book. From the finite additivity of m_s and m, we then have that m_s and m agree on the algebra $\mathcal{A}$ of finite disjoint unions of h-intervals. By the uniqueness part of Theorem 1.14 and the fact that $(m, \mathcal{A})$ is σ-finite, we then have that m_s and m agree on $\mathcal{B}_{\mathbb{R}}$.

Now we can see that the domain of m_s includes all of $\mathcal{L}$. Indeed, for each $E \in \mathcal{L}$, we have (since the measurable sets are the completion of $\mathcal{B}_{\mathbb{R}}$) that $E = F \cup N$ where $F \in \mathcal{B}_{\mathbb{R}}$, $N \subset N'$, $N' \in \mathcal{B}_{\mathbb{R}}$ and $m(N') = 0$. Then $E + s = (F + s) \cup (N + s)$, and $(F + s), (N' + s) \in \mathcal{B}_{\mathbb{R}} + s = \mathcal{B}_{\mathbb{R}}$ and $m(N' + s) = m_s(N') = m(N') = 0$ (thanks to the previous paragraph), and so we may conclude that $E + s \in \mathcal{L}$.

Finally, to see that m_s and m agree on $\mathcal{L}$, it suffices by Theorem 1.9 (since m and m_s agree on $\mathcal{B}_{\mathbb{R}}$ and m is the unique extension of $(m, \mathcal{B}_{\mathbb{R}})$ to a measure on $\mathcal{L}$) to note that, as in the third paragraph, m_s is a measure on $\mathcal{L}$.

$\square$