1.5 #29 Let E be a Lebesgue measurable set.

(a) If $E \subset N$ where N is the non measurable set described in section 1.1, then $m(E) = 0$

(b) If $m(E) > 0$, then E contains a non measurable set. (It suffices to assume $E \subset [0,1]$. In the notation of section 1.1, $E = \bigcup_{r\in R} E \cap N_r$)

Proof. Starting with (a), suppose that $E \subset N$ is measurable. For each number $r \in R := \mathbb{Q} \cap [0,1)$ write

$$E_r := ((E \cap [0,1-r)) + r) \cup (E \cap [1-r,1) + r - 1).$$

Then we have by Theorem 1.21 and additivity

$$\begin{aligned} m(E) &= m(E \cap [0,1-r)) + m(E \cap [1-r,1)) \\ &= m((E \cap [0,1-r)) + r) + m(E \cap [1-r,1) + r - 1) \\ &= m(E_r) \end{aligned}$$

and that E_r is measurable. Furthermore, since $E_r \subset N_r$ and $\bigcup_{r\in R} N_r = [0,1)$ is a disjoint union, we have that

$$\bigcup_{r\in R} E_r \subset [0,1)$$

and the union on the left above is disjoint. Therefore

$$\sum_{r\in R} m(E) = \sum_{r\in R} m(E_r) \leq m([0,1)) = 1$$

and since R is infinite, we must have $m(E) = 0$.

For (b), we first introduce the notation

$$G^r := ((G \cap [0,r)) + 1 - r) \cup ((G \cap [r,1)) - r)$$

for $G \subset [0,1)$ and $r \in R$. Note that $(N_r)^r = N$ and that if G is measurable then G^r is measurable with $m(G^r) = m(G)$. Suppose that $m(E) > 0$. Then

$$m(E) = \sum_{n\in\mathbb{Z}} m(E \cap [n, n+1))$$

and so for some $n_0 \in \mathbb{Z}, m(E \cap [n_0, n_0 + 1) > 0$. Write $F = (E \cap [n_0, n_0 + 1) - n_0$ so that $F \subset [0,1)$ and $m(F) > 0$. Again using the fact that $\bigcup_{r\in R} N_r = [0,1)$ is a disjoint union, we would have

$$m(F) = \sum_{r\in R} m(F \cap N_r)$$

if $m(F \cap N_r)$ was measurable for every $r \in R$. If that was the case, then for every $r \in R$ we would also have $(F \cap N_r)^r \subset N$ measurable and so $m(F \cap N_r) = m((F \cap N_r)^r) = 0$ by part (a), giving $m(F) = 0$. Therefore, since $m(F) > 0$ we must have $F \cap N_r$ non measurable for some $r \in R$ and so $(F \cap N_r) + n_0 \subset E$ is non-measurable. $\square$