5.2 #17 A linear functional f on a normed vector space is bounded iff $f^{-1}(\{0\})$ is closed

Proof. Let $K = f^{-1}(\{0\})$. Then if f is bounded, f is continuous, so K must be closed since $\{0\}$ is closed. For the other direction, suppose that f is not bounded. Then (by definition of boundedness) for each n there is an $x_n \neq 0$ with $|f(x_n)| \geq n\|x_n\|$. Letting $z_n = x_1 - f(x_1)\frac{x_n}{f(x_n)}$ we have $\{z_n\}_{n=1}^{\infty} \subset K$ and $\|z_n - x_1\| = |f(x_1)|\frac{\|x_n\|}{|f(x_n)|} \leq |f(x_1)|\frac{1}{n}$ so $x_1 = \lim_{n\to\infty} z_n$. But $x_1 \notin K$ so K is not closed. $\square$

Bonus Problem A Banach space is a Hilbert space if and only if its norm satisfies the parallelogram law.

Proof. We know every Hilbert space satisfies the parallelogram law:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2. \tag{1}$$

For the other direction, suppose a Banach space satisfies (1) and set

$$\langle x, y\rangle := \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right). \tag{2}$$

Then, it follows from homogeneity of $\|\cdot\|$ that $\langle x, x\rangle = \|x\|$, so it suffices to check that $\langle\cdot,\cdot\rangle$ is an inner product. Positive definiteness is immediate from positive definiteness of $\|\cdot\|$. Conjugate symmetry follows from homogeneity of $\|\cdot\|$ together with the fact that $y + ix = i(x - iy)$ and $y - ix = -i(x + iy)$. Thus it remains to verify linearity in the first entry. Believe for the moment

$$\langle x_1 + x_2, y\rangle = \langle x_1, y\rangle + \langle x_2, y\rangle. \tag{3}$$

Then, repeated applications of (3) give

$$\langle cx, y\rangle = c\,\langle x, y\rangle \tag{4}$$

whenever c is a nonnegative rational real number. By continuity of $\|\cdot\|$ (i.e. the sub-triangle inequality) we have continuity in x of $\langle x, y\rangle$ and so (4) holds whenever c is a nonnegative real number. Furthermore it follows directly from (2) and homogeneity of $\|\cdot\|$ that (4) holds when $c = -1$ or $c = -i$ and so one more application of (3) gives the general case of (4). Finally, we prove (3) (notice we still haven't used (1), so we'd better use it now). To this end it suffices to show

$$\|x_1+x_2+z\|^2 - \|x_1+x_2-z\|^2 = \|x_1+z\|^2 - \|x_1-z\|^2 + \|x_2+z\|^2 - \|x_2-z\|^2 \tag{5}$$

(apply with $z = y$ and $z = iy$). From two applications of the parallelogram law, the LHS of (5)

$$= 2\|x_1 + \frac{z}{2}\|^2 + 2\|x_2 + \frac{z}{2}\|^2 - 2\|x_1 - \frac{z}{2}\|^2 - 2\|x_2 - \frac{z}{2}\|^2 \tag{6}$$

Applying the parallelogram law two more times, (6)

$$= \quad 4\|\frac{x_1 + z}{2}\|^2 \; - \; 4\|\frac{x_1 - z}{2}\|^2 \; + \; 4\|\frac{x_2 + z}{2}\|^2 \; - \; 4\|\frac{x_2 - z}{2}\|^2$$

which finishes the proof. $\square$