MAA 5617 Midterm

Name: ______________________

Complete two of the following problems (circle the numbers of the two you want graded):

Problem 1 Suppose that $\{B(x_j, r_j)\}_{j=1}^n$ is a finite collection of balls in $\mathbb{R}^d$. Show that there is a subcollection $\{B(x_{j_k}, r_{j_k})\}_{k=1}^l$ of pairwise disjoint balls such that

$$m(\bigcup_{k=1}^{l} B(x_{j_k}, r_{j_k})) \geq 3^{-d}\, m(\bigcup_{j=1}^{n} B(x_j, r_j))$$

where m denotes Lebesgue measure (You are not allowed to trivialize the problem by appealing to a covering lemma in the book).

Solution:
After reordering, we may assume that $r_{j+1} \leq r_j$ for $j = 1, \ldots, n-1$. Let $j_1 = 1$. After $j_1, \ldots, j_s$ have been selected, let

$$j_{s+1} = \min(\{j \in (j_s, n] : B(x_j, r_j) \cap B(x_{j_k}, r_{j_k}) = \emptyset \text{ for } \ k = 1, \ldots, s\})$$

assuming that the set inside the minimum is nonempty (otherwise terminate the process and set $l = s$).

The balls $\{B(x_{j_k}, r_{j_k})\}_{k=1}^l$ are clearly pairwise disjoint by construction. Also, for each j not selected we have $B(x_j, r_j) \cap B(x_{j_k}, r_{j_k}) \neq \emptyset$ for some $j_k < j$. Thus, by the monotonicity of the r_j

$$\bigcup_{k=1}^{l} B(x_{j_k}, 3r_{j_k}) \supset \bigcup_{j=1}^{n} B(x_j, r_j)$$

and so

$$m(\bigcup_{j=1}^{n} B(x_j, r_j)) \leq m(\bigcup_{k=1}^{l} B(x_{j_k}, 3r_{j_k})) \leq \sum_{k=1}^{l} m(B(x_{j_k}, 3r_{j_k}))$$

$$= \sum_{k=1}^{l} 3^d m(B(x_{j_k}, r_{j_k})) = 3^d m(\bigcup_{k=1}^{l} B(x_{j_k}, r_{j_k}).$$

Problem 2 Suppose that $\{T_j\}_{j=1}^{\infty}$ is a sequence of bounded linear transformations from a normed vector space $\mathcal{X}$ into a Banach space $\mathcal{Y}$, suppose $\|T_j\| \leq M < \infty$ for all j and suppose there is a dense set $E \subset \mathcal{X}$ such that $\{T_j(x)\}_{j=1}^{\infty}$ converges for every $x \in E$. Prove that $\{T_j(x)\}_{j=1}^{\infty}$ converges for every $x \in \mathcal{X}$.

Solution: Let $x \in \mathcal{X}$. Since $\mathcal{Y}$ is complete it suffices to show that $\{T_j(x)\}_{j=1}^{\infty}$ is Cauchy. Let $\epsilon > 0$, we need to find N so that $\|T_m(x) - T_n(x)\| < \epsilon$ for all $m, n \geq N$. Since E is dense in X, we can find an $x' \in E$ with $\|x - x'\| < \frac{\epsilon}{3M}$. Since $\{T_j(x')\}_{j=1}^{\infty}$ is convergent, we can find an N so that $\|T_m(x') - T_n(x')\| < \epsilon/3$ for all $m, n \geq N$. Then

$$\begin{aligned}\|T_m(x) - T_n(x)\| &\leq \|T_m(x - x')\| + \|T_m(x') - T_n(x')\| + \|T_n(x' - x)\| \\ &< 2M\|x - x'\| + \frac{\epsilon}{3} < \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon\end{aligned}$$

for $m, n \geq N$.

Problem 3 Suppose that $\{x_j\}_{j=1}^{\infty}$ is a sequence in a **separable** Hilbert space $\mathcal{X}$ and that $\|x_j\| \leq 1$ for all j. Show that there is a subsequence $\{x_{j_k}\}_{k=1}^{\infty}$ such that for every $y \in \mathcal{X}$, $\{\langle y, x_{j_k}\rangle\}_{k=1}^{\infty}$ is convergent (*Hint:* you are allowed to use the result of Problem 2 even if you didn't attempt it).

Solution: Since $\mathcal{X}$ is separable, it has a countable dense subset $\{y_n\}_{n=1}^{\infty}$. Then (By Cauchy-Schwarz and the fact that $\|x_j\| \leq 1$)

$$\{\langle y_1, x_j, \rangle\}_{j=1}^{\infty} \subset \{z \in \mathbb{C} : |z| \leq \|y_1\|\}.$$

Since the set on the right is compact, there is a convergent subsequence $\{\left\langle y_1, x_{j_{1,k}} \right\rangle\}_{k=1}^{\infty}$. After subsequences $\{x_{j_{l,k}}\}_{k=1}^{\infty}$ have been chosen for $l = 1, \ldots, s$ we choose $\{x_{j_{s+1,k}}\}_{k=1}^{\infty}$ to be a subsequence of $\{x_{j_{s,k}}\}_{k=1}^{\infty}$ such that $\{\left\langle y_{s+1}, x_{j_{s+1,k}} \right\rangle\}_{k=1}^{\infty}$ is convergent (whose existence is again guaranteed by compactness). We then have that the subsequence $\{\left\langle y_n, x_{j_{k,k}} \right\rangle\}_{k=1}^{\infty}$ is convergent for every n.

By the density of $\{y_n\}_{n=1}^{\infty}$ in $\mathcal{X}$ and Problem 2 (with linear operators $T_k(y) := \left\langle y, x_{j_{k,k}} \right\rangle$ and $M = 1$) we have $\{\left\langle y, x_{j_{k,k}} \right\rangle\}_{k=1}^{\infty}$ convergent for all $y \in \mathcal{X}$.