3.4 #26 If λ and μ are positive, mutually singular Borel measures on $\mathbb{R}^n$ and $\lambda + \mu$ is regular, then so are λ and μ

Proof. Finiteness of λ and μ on compact sets is immediate from positivity and the corresponding property for $\lambda + \mu$.

Let E be a Borel set. By symmetry, it suffices to show that

$$\lambda(E) = \inf\{\lambda(U) : E \subset U, U \text{ open}\}. \tag{1}$$

Let $G_k = [-k, k]^n$ so that $\mathbb{R}^n = \bigcup_k G_k$ and $(\lambda + \mu)(G_k) < \infty$. Let $\epsilon > 0$. By the outer regularity of $\lambda + \mu$ we can find, for each k, an open $U_k \supset (E \cap G_k)$ with $(\lambda + \mu)(U_k \setminus (E \cap G_k)) < \epsilon 2^{-k}$.

Letting $U = \bigcup_k U_k$, we then have $(\lambda+\mu)(U \setminus E) \leq \epsilon$ and so $\lambda(U \setminus E) < \epsilon$, which gives (1) since ϵ was arbitrary.

It seems we didn't need to use mutual singularity??

□