1.5 #33 There exists a Borel set $A \subset [0,1]$ such that $0 < m(A \cap I) < m(I)$ for for every subinterval I of $[0,1]$. (Hint: Every subinterval of $[0,1]$ contains Cantor-type sets of positive measure).

Proof. Here is Rudin's answer or for more discussion stackexchange. □

3.5 #41 Let $A \subset [0,1]$ be a Borel set such that $0 < m(A \cap I) < m(I)$ for every subinterval I of $[0,1]$ (Exercise 33, Chapter 1).

(a) Let $f(x) = m([0,x] \cap A)$. Then F is absolutely continuous and strictly increasing on $[0,1]$, but $F' = 0$ on a set of positive measure.
(b) Let $g(x) = m([0,x] \cap A) - m([0,x] \setminus A)$. Then g is absolutely continuous on $[0,1]$, but g is not monotone on any subinterval of $[0,1]$.

Proof. For (a): It is clear that f is strictly increasing, since $m(A \cap I) > 0$ for every I. Absolute continuity is also immediate since whenever $x < y$ we have $f(y) - f(x) < m((x,y]) = x - y$. (i.e. we may take $\delta = \epsilon$). By theorem 3.35 we have

$$f(x) = \int_0^x f'(y)\ dy$$

for $x \geq 0$. But, we also have $f(x) = \int_0^x 1_A(y)\ dy$ for $x \geq 0$. Thus, we have $f' = 1_A$ almost everywhere (since $f'dm$ and $1_A dm$ agree on intervals, they must agree on all sets by uniqueness of outer measures). But $m(A) < m([0,1])$ and so $1_A = 0$ on a positive measure subset of $[0,1]$.

For (b): Let $h(x) = m([0,x] \cap A^c)$. Since A^c satisfies the hypotheses of the problem, we also have h absolutely continuous with $h' = 1_{A^c}$ almost everywhere. Then $g = f - h$ is absolutely continuous and $g' = 1_A - 1_{A_c}$ almost everywhere. Since every interval contains a positive measure subset of points in A and A^c we have $g' > 0$ and $g' < 0$ on points in every interval, which would be impossible if g was monotone on an interval. □