5.1 #7 Let $\mathcal{X}$ be a Banach space

(a) If $T \in L(\mathcal{X}, \mathcal{X})$ and $\|I - T\| < 1$ where I is the identity operator, then T is invertible; in fact, the series $\sum_{n=0}^{\infty}(I-T)^n$ converges in $L(\mathcal{X}, \mathcal{X})$ to T^{-1}.

(b) If $T \in L(\mathcal{X}, \mathcal{X})$ is invertible and $\|S - T\| < \|T^{-1}\|^{-1}$, then S is invertible. Thus, the set of invertible operators is open in $L(\mathcal{X}, \mathcal{X})$.

Proof. For (a), first note that multiplication by a fixed vector in a Banach Algebra is a continuous map, i.e. if $S = \lim_{n\to\infty} S_n$ we have

$$\lim_{n\to\infty} \|TS - TS_n\| = \lim_{n\to\infty} \|T(S - S_n)\| \leq \|T\| \lim_{n\to\infty} \|S - S_n\| = 0.$$

Now, observe that since $\|I - T\| < 1$ we have

$$\sum_{n=0}^{\infty} \|(I-T)^n\| \leq \sum_{n=0}^{\infty} \|I - T\|^n = \frac{1}{1 - \|I - T\|} < \infty$$

and so the series $\sum_{n=0}^{\infty}(I-T)^n$, being absolutely convergent, must be convergent. Noting that

$$T\sum_{n=0}^{m}(I-T)^n = (I - (I-T))\sum_{n=0}^{m}(I-T)^n = I - (I-T)^{m+1}$$

(the "geometric series trick"), we then have, by continuity,

$$T\sum_{n=0}^{\infty}(I-T)^n = \lim_{n\to\infty} I - (I-T)^{m+1} = I$$

(in the last identity we are again using $\|I - T\| < 1$). By an analogous argument

$$\left(\sum_{n=0}^{\infty}(I-T)^n\right) T = I$$

and so $\sum_{n=0}^{\infty}(I-T)^n = T^{-1}$.

For (b) : Write

$$S = (T - (T - S)) = T(I - T^{-1}(T - S))$$

Since $\|T^{-1}(T-S)\| < 1$ we have $I - T^{-1}(T-S)$ invertible by (a). Thus since T is assumed to be invertible (and one can check in the usual way that the set of invertible elements is closed under multiplication) we have S invertible. $\square$