5.2 #20 If $\mathcal{M}$ is a finite-dimensional subspace of a normed vector space $\mathcal{X}$, there is a closed subspace $\mathcal{N}$ such that $\mathcal{M} \cap \mathcal{N} = \{0\}$ and $\mathcal{M} + \mathcal{N} = \mathcal{X}$

Proof. Since $\mathcal{M}$ is finite-dimensional, it has some basis $\{e_i\}_{i=1}^n$. For each $i = 1, \ldots, n$ we may apply the Hahn-Banach theorem (as in Theorem 5.8a) to find a bounded linear functional f_i on $\mathcal{X}$ such that $f_i(e_i) = 1$ and $f_i(e_j) = 0$ for $j \neq i$ (Specifically we construct f_i to vanish on $\mathrm{span}(\{e_j : j \neq i\})$ and be nonzero on e_i and renormalize appropriately. Note that the subspace is closed since it is finite dimensional).

Then, for any $x \in \mathcal{M}$ one can check

$$x = \sum_{i=1}^{n} f_i(x) x_i. \tag{1}$$

Let

$$\mathcal{N} = \bigcap_{i=1}^{n} f_i^{-1}(\{0\}).$$

Then $\mathcal{N}$ is a subspace of $\mathcal{X}$ due the the linearity of the f_i's. And $\mathcal{M}$ is closed since it is the intersection of closed (by the boundedness of the f_i's) sets. Furthermore, by (1) we have $\mathcal{M} \cap \mathcal{N} = \emptyset$.

Given any $x \in X$ write

$$x_{\mathcal{M}} = \sum_{i=1}^{n} f_i(x) x_i$$

and $x_{\mathcal{N}} = x - x_{\mathcal{M}}$. Then $x = x_{\mathcal{M}} + x_{\mathcal{N}}$ and $x_{\mathcal{M}} \in \mathcal{M}$. Also for each j

$$\begin{aligned} f_j(x_{\mathcal{N}}) = f_j(x) - f_j(x_{\mathcal{M}}) &= f_j(x) - \sum_{i=1}^{n} f_i(x) f_j(x_i) \\ &= f_j(x) - f_j(x) = 0 \end{aligned}$$

and so $x_{\mathcal{N}} \in \mathcal{N}$. Therefore $\mathcal{X} = \mathcal{M} + \mathcal{N}$.

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