5.3 #37 Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. If $T : \mathcal{X} \to \mathcal{Y}$ is a linear map such that $f \circ T \in \mathcal{X}^*$ for every $f \in \mathcal{Y}^*$, then T is bounded.

Proof. By the closed graph theorem, it suffices to show that whenever $x_n \to x$ and $T(x_n) \to y$ then $y = T(x)$. Suppose not. Then, since (by some corollary of the Hahn-Banach theorem) $\mathcal{Y}^*$ separates points in Y there is an $f \in \mathcal{Y}^*$ such that $f(T(x)) \neq f(y)$. On one hand, by continuity of f

$$\lim_{n\to\infty} f(T(x_n)) = f(y)$$

On the other hand by continuity of $f \circ T$

$$\lim_{n\to\infty} f(T(x_n)) = f(T(x)).$$

But we can't have both (by choice of f), so contradiction.

Alternatively:

Let $A = \{y \in \mathcal{Y}^* : \|y\| = 1\}$. Then for each $x \in \mathcal{X}$

$$\sup_{f\in A} |f \circ T(x)| \leq \|T(x)\| < \infty.$$

Since, by assumption, the $f \circ T \in \mathcal{X}^*$, the uniform boundedness principle gives a C such that

$$\sup_{f\in\mathcal{A}} \|f \circ T\| \leq C. \tag{1}$$

By the Hahn-Banach theorem, for each $x \in \mathcal{X}$ there is an $f \in A$ such that $|f(T(x))| = \|T(x)\|$. However, by (1) we have $|f(T(x))| \leq C\|x\|$ and so $\|T(x)\| \leq C\|x\|$. Thus, T is bounded.

□