MAA 5617 Midterm

Name: ______________________

Complete two of the following problems (circle the numbers of the two you want graded):

Problem 1 Suppose that ν and μ are positive measures on $(\mathcal{X}, \mathcal{M})$, that $\nu \ll \mu$, and that ν is finite. Show that for every $\epsilon > 0$ there is a $\delta > 0$ such that for all $E \in \mathcal{M}$ with $\mu(E) < \delta$, we have $\nu(E) < \epsilon$. (You are not allow to use any theorem that trivializes the problem (for example you can't use the Radon-Nikodym theorem)).

Solution:

Suppose not. Then there is an $\epsilon > 0$ such that for every $\delta > 0$ there exists a set E with $\mu(E) < \delta$ but $\nu(E) \geq \epsilon$. Applying this with $\delta = 2^{-j}$ we obtain sets $\{E_j\}_{j=1}^{\infty}$. Letting $F_k = \cup_{j=k}^{\infty} E_j$ we have $\mu(F_k) \leq 2^{1-k}$ by countable subadditivity, so by monotonicity $\mu(\cap_{k=1}^{\infty} F_k) = 0$. On the other hand (since ν is finite) continuity from above gives

$$\nu(\cap_{k=1}^{\infty} F_k) = \lim_{k \to \infty} \nu(F_k) \geq \epsilon.$$

Problem 2 Suppose that $\{f_j\}_{j=1}^{\infty}$ is a sequence of functions on $\mathbb{R}$ that are of normalized bounded variation. Show that if $\lim_{j\to\infty} f_j = f$ pointwise then

$$T_f(x) \leq \liminf_{j\to\infty} T_{f_j}(x).$$

Solution:

Let $\epsilon > 0$ and choose $x_0 < \ldots < x_n = x$ so that

$$\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \geq T_f(x) - \epsilon/2$$

Since $f_j \to f$ pointwise, we can find M so that for $j \geq M$ and $k = 0, \ldots, n$ we have $|f_j(x_k) - f(x_k)| < \epsilon/(4n)$. Then by the sub-triangle and triangle inequalities

$$\begin{aligned} ||f(x_k) - f(x_{k-1})| - |f_j(x_k) - f_j(x_{k-1})|| &\leq |f(x_k) - f_j(x_k) + f(x_{k-1}) - f_j(x_{k-1})| \\ &< \epsilon/(4n) + \epsilon/(4n) \\ &= \epsilon/(2n) \end{aligned}$$

and in particular

$$T_{f_j}(x) \geq \sum_{k=1}^{n} |f_j(x_k) - f_j(x_{k-1})| > \sum_{k=1}^{n} (|f(x_k) - f(x_{k-1})| - \epsilon/(2n)) \geq T_f(x) - \epsilon.$$

So

$$\inf_{j\geq M} T_{f_j}(x)) \geq T_f(x) - \epsilon$$

and thus

$$\liminf_{j\to\infty} T_{f_j}(x) \geq T_f(x) - \epsilon.$$

Since ϵ was arbitrary, we are done.

Problem 3 Suppose that X is a normed vector space and that M is a closed subspace of X with $M \neq X$. Show that there is an $x \in X$ with $x \neq 0$ and

$$\inf_{y \in M} \|x - y\| \geq \frac{1}{2}\|x\|.$$

Solution:

Since $M \neq X$ we may find a $z \in X \setminus M$. Let

$$\delta = \inf_{y \in M} \|z - y\|$$

and choose a $y_0 \in M$ with $\|z - y_0\| \leq 2\delta$. Letting $x = z - y_0$ we have

$$\|x - y\| = \|z - (y_0 + y)\| \geq \delta \geq \|x\|/2$$

for every $y \in M$.