3.2 #14 If ν is an arbitrary signed measure and μ is a σ-finite measure on $(\mathcal{X}, \mathcal{M})$ such that $\nu \ll \mu$, there exists an extended μ-integrable function $f : \mathcal{X} \to [-\infty, \infty]$ such that $d\nu = f d\mu$. Hints:

(1) It suffices to assume that μ is finite and ν is positive
(2) With these assumptions, there exists $E \in \mathcal{M}$ that is σ-finite for ν such that $\mu(E) \geq \mu(F)$ for all sets F that are σ-finite for ν
(3) The Radon-Nikodym theorem applies on E. If $F \cap E = \emptyset$, then either $\nu(F) = \mu(F) = 0$ or $\mu(F) > 0$ and $\nu(F) = \infty$

Proof. 1.)Let $\nu = \nu^+ - \nu^-$ be the Jordan decomposition for ν. By Exercise 8 we have that ν^+ and $\nu^- \ll \mu$. Thus, by applying the result for positive measures we have $d\nu = f^+ dm - f^- dm$. It follows from the definition of signed measures that either ν^+ or $\nu-$ is finite, and so we must have either f^+ or f^- integrable, and so $f := f^+ - f^-$ is extended-integrable, so fdm makes sense and is equal to $d\nu$.

Henceforth, assume we are working with positive ν.

Since μ is σ-finite, there is a countable cover $\{E_j\}_j \subset \mathcal{M}$ for $\mathcal{X}$ with $\mu(E_j) < \infty$ for every j. After the disjointification trick, we may assume that the E_j are pairwise disjoint. Let μ_j be the restriction of μ to E_j (i.e. $\mu_j(E) := \mu(E \cap E_j)$ for $E \in \mathcal{M}$) and let ν_j be the restriction of ν to E_j. Clearly $\nu_j \ll \mu_j$. So, by the theorem for finite measures, we have $dv_j = f_j dm_j$ and hence (since the f_j are positive, no problem integrating the infinite sums) letting $f = \sum_j f_j$ we have (by disjointness of the E_j) $d\nu = fdm$.

Henceforth assume that μ is finite.

2.) Let $\alpha = \sup\{\mu(F) : F \text{ is } \sigma-\text{finite for } \nu\}$, and choose a sequence of ν-σ-finite sets F_j with $\alpha = \lim_{j\to\infty} \mu(F_j)$. Letting $E = \bigcup_j F_j$ we have E σ-finite for ν and $\mu(E) = \alpha$ (by positivity of μ).

3.) Suppose $F \cap E = \emptyset$. If $\mu(F) = 0$ then by absolute continuity $\nu(F) = 0$. Otherwise $\mu(F) > 0$ and so (by finiteness of μ) $\mu(F \cup E) > \mu(E)$. By maximality of E, this implies $F \cup E$ is not σ-finite for ν and hence (since the union of a finite set with a σ-finite set is σ-finite) $\nu(F) = \infty$.

Now, let ν_E and μ_E be the restrictions of ν and μ to E. Applying the Radon-Nikodym theorem $\nu_E = f_E d\mu_E$ for some f_E. Defining f by $f(x) = f_E(x)$ if $x \in E$ and $f(x) = \infty$ if $x \notin E$ we have $d\nu = fd\mu$. Indeed:

$$\nu(G) = \nu(G \cap E) + \nu(G \cap E^c) = \int_{G\cap E} f d\mu + \int_{G\cap E^c} f d\mu = \int_G f d\mu$$

where the penultimate identity holds from the observation in the previous paragraph and the fact that the integral of ∞ over a set of measure 0 is 0 and the integral of ∞ over a set of positive measure is ∞. $\square$