THE (d, k) KAKEYA PROBLEM AND ESTIMATES FOR THE X-RAY TRANSFORM

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Abstract

A (d, k) set is a subset of $\mathbb{R}^d$ containing a translate of every k-dimensional disc of diameter 1. We show that if $(1+\sqrt{2})^{k-1} + k > d$ and $k \geq 2$, then every (d, k) set has positive Lebesgue measure. This improves a result of Bourgain, who showed that the analogous statement holds when $2^{k-1}+k \geq d$ and $k \geq 2$. We obtain this improvement in two parts. First, we replace Bourgain's main estimate with a simple recursive maximal operator bound involving mixed-norm estimates for the X-ray transform. This method allows us to simplify Bourgain's proof, allows us to obtain improved bounds for the maximal operator associated with (d, k) sets, and demonstrates that improved estimates for (d, k) sets would follow from new bounds for the X-ray transform. Second, we adapt arithmetic-combinatorial methods of Katz and Tao to obtain improved bounds for the X-ray transform suitable for use with the recursive maximal operator bound.

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This thesis is dedicated to Christina.

Contents

Chapter 1

Introduction

1.1 History

We begin by giving a brief history of the Kakeya problem, mentioning only the work which is perhaps most relevant to the specific problems considered in this document. For complete surveys, see [39], [45], [41], and [7].

In 1917 Kakeya posed the question: What is the minimum area of a subset of $\mathbb{R}^2$ in which a unit line segment can be rotated 180 degrees? Independently, in 1919, Besicovitch constructed a measure zero subset of $\mathbb{R}^2$ containing a line segment in every direction for use as a counterexample in a problem of Riemann integration. After leaving Russia, Besicovitch learned of Kakeya's question and used a variant of his construction to show that Kakeya's sets may be taken to have arbitrarily small measure [3],[4].

Today, we say that a subset of $\mathbb{R}^d$ is a Kakeya set if it contains a unit line segment in every direction, and we say that a Kakeya set is a Besicovitch set if it has Lebesgue measure zero. The standard construction of Besicovitch sets in $\mathbb{R}^2$ is due to Perron [34]. This construction starts with the triangle whose vertices are $(0,0), (0,1), (\sqrt{2}, 0)$, which contains unit line segments in $\frac{1}{4}$ of all directions but has relatively large area. Perron observed that by cutting the triangle into small angular sectors and carefully translating the sectors along the x-axis so that they have large overlap, the resulting set still contains line segments in $\frac{1}{4}$ of all directions but may be taken to have arbitrarily small area. A limiting argument then yields $\frac{1}{4}$ of a Besicovitch set, and the union of 4 rotations gives the full Besicovitch set; for details of the proof, see [38]. The Cartesian product of a Besicovitch set in $\mathbb{R}^2$ with the unit ball in $\mathbb{R}^{d-2}$ then gives a Besicovitch set in $\mathbb{R}^d$.

Once we know that there are Kakeya sets of measure zero, it is natural to ask whether Kakeya sets must even have full dimension. The weakest formulation of this question is in terms of Minkowski dimension. Given a bounded subset E of $\mathbb{R}^d$ and $\delta > 0$, let E_δ denote the δ-neighborhood of E. Then E is said to have upper Minkowski dimension at least η if for every $\epsilon > 0$

$$\limsup_{\delta \to 0} \frac{\mathcal{L}^d(E_\delta)}{\delta^{d-(\eta-\epsilon)}} > 0,$$

where $\mathcal{L}^d$ denotes Lebesgue measure in $\mathbb{R}^d$. It is conjectured that every Kakeya set in $\mathbb{R}^d$ has upper Minkowski dimension d, and stronger still that every Kakeya set has Hausdorff dimension d. This was shown to be the case when $d = 2$ by Davies in [12], also see [11] and [21], but is still far from being resolved in higher dimensions.

When $d > 2$, we may also consider the planar generalizations of Kakeya sets. Let $G(d, k)$ denote the set of k-dimensional linear subspaces of $\mathbb{R}^d$. We say that a subset E of $\mathbb{R}^d$ is a (d, k) set if for every $L \in G(d, k)$, E contains a translate of the intersection of L with the ball centered at the origin with radius $\frac{1}{2}$, $B(0, \frac{1}{2})$. Marstrand showed in [24] that every $(3, 2)$ set has positive measure, in other words that there are no Besicovitch $(3, 2)$ sets. This contrast between the case $k = 2$ and $k = 1$ may be justified heuristically by counting parameters. The dimension of $G(d, k)$ is $k(d-k)$ and the dimension of a k-plane is k, so one might expect the dimension of a (d, k) set to be $\min(d, k(d-k)+k)$. When $k = 1$, $k(d-k)+k = d$; however when $k > 1$, $k(d-k)+k$ is strictly larger than d. Thus, $(d, 1)$ sets should “barely” have full dimension and we might expect them to have zero measure, whereas $(d, 2)$ sets should "easily" have full dimension and thus might be expected to have positive measure. Although parameter counting clearly works in the case of $(2, 1)$ sets and $(3, 2)$ sets, it is known to fail in certain cases when one replaces $G(d, k)$ by a lower dimensional subset of $G(d, k)$, or when one replaces the k-planes by lower dimensional subsets of k-planes; see [45], [35], [31], [32]. Nonetheless, it is conjectured that (d, k) sets have positive measure when $k \geq 2$, and soon after Marstrand's work this was shown to be the case when $k > \frac{d}{2}$ by Falconer [14] (the papers [15] and later [26] are erroneous).

Direct interest in the Kakeya problem was resparked by Bourgain's revolutionary paper [5]. For $L \in G(d, k)$, $\delta > 0$, and $a \in \mathbb{R}^d$, let $L_\delta(a)$ denote the δ-neighborhood of $a + (L \cap B(0, \frac{1}{2}))$. The Kakeya maximal operator for k-planes is defined

$$\mathcal{M}_\delta^k[f](L) = \sup_{a \in \mathbb{R}^d} \frac{1}{\mathcal{L}^d(L_\delta(a))} \int_{L_\delta(a)} |f(x)| \, dx.$$

Bourgain observed that L^p bounds for $\mathcal{M}_\delta^1$ imply Hausdorff dimension estimates for $(d, 1)$ sets. Specifically, the bound [1]

$$\|\mathcal{M}_\delta^1[f]\|_{L^1(G(d,1))} \lesssim \delta^{-\frac{\alpha}{p}} \|f\|_{L^p(\mathbb{R}^d)}$$

for some $\alpha > 0$ implies that $(d, 1)$ sets have Hausdorff dimension at least $d - \alpha$. Although not explicit in [5], the same method gives the corresponding result when $k > 1$. From Bourgain's observation, Drury's bound [13] for the X-ray transform, and Christ's bound [10] for the k-plane transform, it follows that (d, k) sets must have Hausdorff dimension at least $\frac{k(d+1)}{k+1}$ for every d, k. Bourgain also gave an improved bound for $\mathcal{M}_\delta^1$, implying that $(3, 1)$ sets have dimension at least $\frac{7}{3}$ and $(d, 1)$ sets have dimension at least $\frac{d+1}{2} + \epsilon_d$

[1] We will use $\cdot \lesssim \cdot$ to denote $\cdot \leq C \cdot$ where C may depend, for example, on $d, k, p, q, \epsilon, \alpha$ but not on f, δ, λ. A similar notation $\lessapprox$ will be used and explained later.

for $d > 3$, where $\epsilon_d > 0$ is defined recursively, also see [36]. By combining this improved bound with the L^2 estimate [37] and a recursive metric-entropy estimate, he then showed that there are no (d, k) Besicovitch sets when $2^{k-1} + k \geq d$ and $k \geq 2$. In the cases $(d, k) = (4, 2), (7, 3)$ where the recursive metric-entropy estimate was not needed, he also gave L^p bounds for the maximal operator

$$\mathcal{M}^k[f](L) = \sup_{a \in \mathbb{R}^d} \int_{L+a} |f(x)| \, dx,$$

showing that, for f supported on a fixed ball,

$$\|\mathcal{M}^k[f]\|_{L^p(G(d,k))} \lesssim \|f\|_{L^p(\mathbb{R}^d)},$$

for $p > 2$ when $(d, k) = (4, 2)$ and for $p > 3$ when $(d, k) = (7, 3)$. He alluded to a bound when $2^{k-1} + k \geq d$, but it is clear that this bound would have to be for a very large p due to the inefficiencies of the metric-entropy estimates.

Bourgain's bounds for $\mathcal{M}^1_\delta$ were later surpassed by Wolff [43], who gave a bound implying that $(d, 1)$ sets have Hausdorff dimension at least $\frac{d+2}{2}$. Wolff's "hairbrush" argument can be viewed as a unification of Bourgain's "bush" argument from [5] with the L^2 method of Cordoba [11]. The hairbrush argument was later refined by Łaba, Tao, and Wolff to give mixed-norm estimates for the X-ray transform in $\mathbb{R}^3$ [44] and $\mathbb{R}^d$, $d > 3$ [23].

The bush and hairbrush arguments, and work on the Kakeya problem for circles, are largely incidence-combinatorial in nature. The next breakthrough came with Bourgain's use of methods from arithmetic combinatorics. While working on quantitative estimates related to Szemerédi's theorem, Gowers [16] developed a quantitative version of the Balog-Szemerédi theorem [2] which relates the size of sum-sets and difference-sets.

Bourgain applied this result to the Kakeya problem [6], showing that $(d, 1)$ sets have lower Minkowski dimension at least $\frac{13d+12}{25}$, and giving a less substantial improvement for the maximal operator bound. Katz and Tao then gave related arithmetic estimates [19], [17], which were more specialized to the Kakeya problem and hence more elementary and more flexible for use in maximal operator and Hausdorff dimension estimates than those of Balog-Szemerédi type. With these estimates they were able to give improved maximal operator bounds and show that $(d, 1)$ sets have lower Minkowski dimension at least $\frac{d-1}{\beta} + 1$ where $\beta = 1.675\ldots$ is the largest root of $\beta^3 - 4\beta + 2 = 0$.

Katz and Tao also combined their arithmetic estimates with hairbrush type improvements to show that the Hausdorff dimension of $(d, 1)$ sets is at least $(2 - \sqrt{2})(d - 4) + 3$. Łaba, Katz, and Tao combined arithmetic methods with the hairbrush argument and an intricate "sticky/plainy/grainy" analysis [18] [22] to obtain marginal improvements for the upper Minkowski dimension over Wolff's estimate in dimensions 3 and 4.

As we will discuss in Chapter 4, the Kakeya problem may also be considered in the setting of vector spaces over finite-fields. This variant of the problem first appeared in the literature in [45]. Many of the methods used in Euclidean space transfer to finite fields with the most notable exception being the sticky/plainy/grainy analysis of Katz, Łaba, and Tao. Interestingly, alternative methods have been employed to obtain analogous results in 3 and 4 dimensions [8], [42]. It remains to be seen whether these methods can be adapted for use in Euclidean space.

Recent work on the (d, k) Kakeya problem for $k > 1$ has focused on bush and hairbrush type arguments. Alvarez used the bush argument to reprove, except for endpoints, the maximal operator bound implied by [10] when $k = 2$, and used the hairbrush argument to to show that the lower Minkowski dimension of $(d, 2)$ sets is at least $\frac{2d+3}{3}$ and to give estimates for a Kakeya-type maximal operator for lines in $\mathbb{C}^d$, see [1]. Mitsis improved this Minkowski dimension estimate to a Hausdorff dimension estimate [27], also see [28]. In the setting of finite fields, Bueti improved this Hausdorff dimension estimate to a maximal operator bound and then generalized this bound from $k = 2$ to $k < d - 1$, see [9].

1.2 Statement of main results

1.2.1 Maximal operators associated with k-planes

Our work on the (d, k) Kakeya problem begins by revisiting, as suggested by A. Seeger, Bourgain's recursive metric-entropy estimate from [5]. Roughly stated, Bourgain showed that if $(d - 1, k - 1)$ sets always have codimension less than α, then (d, k) sets must always have codimension less than $\frac{\alpha}{2}$. Our initial goal was to adapt the geometric content of Bourgain's proof in a manner which was more efficient for L^p bounds of $\mathcal{M}^k_\delta$ and $\mathcal{M}^k$. A reasonably efficient "recursive maximal operator bound" was thus obtained in the unpublished work [33]. Upon completion of [33], the author realized that the recursive maximal operator bound was essentially Drury's bound [13] for the X-ray transform combined with Proposition 2.5 below and an interpolation.

The X-ray transform is the case $k = 1$ of the k-plane transform

$$T^k[f](L, x) = \int_{L+x} f(y)dy$$

where f is a suitable function on $\mathbb{R}^d$, $L \in G(d, k)$, and x is in the orthogonal complement $L^\perp$ of L. We are interested in mixed-norm estimates

$$\|T^k[f]\|_{L^q(L^r)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$$

where

$$\|T^k[f]\|_{L^q(L^r)} = \left(\int_{G(d,k)} \left(\int_{L^\perp} |T^k[f](L,x)|^r \, dx \right)^{\frac{q}{r}} \, dL \right)^{\frac{1}{q}}, \tag{1.1}$$

and where we only consider functions supported on a fixed ball. Above, integration on $G(d,k)$ is with respect to rotation invariant measure, see Chapter 2.

Proposition 2.5[2]**.** *Suppose the bounds*

$$\|\mathcal{M}_\delta^{k-1}[f]\|_{L^{q_0}(G(d-1,k-1))} \lesssim \delta^{-\frac{\alpha_0}{p_0}} \|f\|_{L^{p_0}(\mathbb{R}^{d-1})} \tag{1.2}$$

and

$$\|T^1[g]\|_{L^{q_0}(L^{p_0})} \lesssim \|g\|_{L^{p_1}(\mathbb{R}^d)}$$

are known to hold for $f \in L^{p_0}(\mathbb{R}^{d-1})$ and for $g \in L^{p_1}(\mathbb{R}^d)$ supported on a fixed ball, where $q_0 \geq p_1$. Then the bound

$$\|\mathcal{M}_\delta^k[f]\|_{L^{q_0}(G(d,k))} \lesssim \delta^{-\frac{\alpha_1}{p_1}} \|f\|_{L^{p_1}(\mathbb{R}^d)}$$

holds for $f \in L^{p_1}(\mathbb{R}^d)$, where $\alpha_1 = \alpha_0 \frac{p_1}{p_0}$.

An analogous statement holds with $\mathcal{M}_\delta^{k-1}$ replaced by T^{k-1}. Heuristically, Proposition 2.5 says that if the X-ray transform in $\mathbb{R}^d$ is known to be bounded from $L^p \to L^q(L^r)$, then when passing from $\mathcal{M}_\delta^{k-1}$ in $\mathbb{R}^{d-1}$ to $\mathcal{M}_\delta^k$ in $\mathbb{R}^d$, we may improve α by a factor of $\frac{p}{r}$. Recalling from Section 1.1 that a bound (1.2) implies that $(d-1,k-1)$ sets must have co-Hausdorff-dimension less than α_0, we see that our codimension estimate is thus improved by a factor of $\frac{p}{r}$ when passing from $(d-1,k-1)$ sets to (d,k) sets. From [10], we know that T^1 is bounded from $L^{\frac{d+1}{2}} \to L^{d+1}(L^{d+1})$ for every d, and so we may take $\frac{p}{r} = \frac{1}{2}$, recovering the dimension estimate given by Bourgain's recursive metric-entropy estimate.

[2]Label numbers corresponding to later chapters refer to propositions and theorems which will be restated for the readers convenience.

A potential difficulty which arises when applying Proposition 2.5 is that the exponent r for the X-ray transform bound must match up with the exponent p_0 for the $\mathcal{M}_\delta^{k-1}$ bound. This difficulty is overcome by interpolating with L^∞ bounds when necessary. However, the interpolation comes at the cost of losing "sharp-p" type estimates, as we will see below. Also, our method does not allow for the optimal range of q. In possible future work we hope to eliminate one or both of these issues.

From a judicious application of Hölder's inequality, one sees that for every f

$$\|\mathcal{M}_\delta^1[f]\|_{L^q(G(d,1))} \lesssim \delta^{-\frac{(d-1)\frac{p}{r}}{p}} \|T^1[f]\|_{L^q(L^r)}, \tag{1.3}$$

and thus from our known bound for the X-ray transform

$$\|\mathcal{M}_\delta^1[f]\|_{L^{d+1}(G(d,1))} \lesssim \delta^{-\frac{d-1}{d+1}} \|f\|_{L^{\frac{d+1}{2}}(\mathbb{R}^d)}.$$

Using this bound as a starting point and recursively applying Proposition 2.5, we obtain

Theorem 2.6. *For* $1 \leq k < d$

$$\|\mathcal{M}_\delta^k[f]\|_{L^q(G(d,k))} \lesssim \delta^{-\frac{\alpha}{p}} \|f\|_{L^p(\mathbb{R}^d)}$$

where $p = \frac{d+1}{2}$, $q = d+1$, *and* $\alpha = \frac{d-k}{2^k}$.

Our primary interest in Theorem 2.6 is that, when combined L^2 methods as in [5], it yields

Theorem 2.4. *Suppose* $2^{k-1} + k > d$. *Then*

$$\|\mathcal{M}^k[f]\|_{L^{\frac{d-1}{2}}(G(d,k))} \lesssim \|f\|_{L^{\frac{d-1}{2}}(\mathbb{R}^d)}$$

for f supported on a fixed ball.

Theorem 2.6 is new when $k > 1$ and $d > 2k$. Theorem 2.4 is new when $k > 3$ and $d > 2k$. It follows from Theorem 2.4 that (d, k) sets must have positive measure when $2^{k-1} + k > d$. This range of k could be improved to match Bourgain's original $2^{k-1} + k \geq d$ by starting with Bourgain's bound for $\mathcal{M}^1_\delta$, and indeed there are now even better bounds to start with, but the statements of the theorems are slightly cleaner as written.

Our approach to proving Theorem 2.4 is, perhaps, interesting for three reasons. First, Proposition 2.5 serves to demystify Bourgain's recursive metric-entropy estimate which was quite technical, whereas the proof of Proposition 2.5 is fairly straightforward. Second, by revealing the connection with the X-ray transform, Proposition 2.5 shows that we may increase our range of k for which $\mathcal{M}^k$ is bounded by proving new mixed-norm estimates for the X-ray transform. Finally, the range of p obtained in Theorem 2.4 is a considerable improvement over what we could have hoped to obtain using the recursive metric-entropy estimate. Unfortunately, this range of p is presumably not optimal, as we conjecture below.

Conjecture 1.1. *Suppose $p < \frac{d}{k}$, $1 \leq k < d$, $k + \frac{d-k}{r} \geq \frac{d}{p}$, and $\frac{k}{q} + \frac{1}{r} \geq \frac{1}{p}$. Then*

$$\|T^k[f]\|_{L^q(L^r)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$$

for f supported on a fixed ball.

Conjecture 1.2. *For $1 \leq k < d, p < \frac{d}{k}, q \leq (d-k)p'$*

$$\|\mathcal{M}^k_\delta[f]\|_{L^q(G(d,k))} \lesssim \delta^{k-\frac{d}{p}} \|f\|_{L^p(\mathbb{R}^d)}. \tag{1.4}$$

Conjecture 1.3. *For $2 \leq k < d, p > \frac{d}{k}, q \leq kp$*

$$\|\mathcal{M}^k[f]\|_{L^q(G(d,k))} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$$

for f supported on a fixed ball.

These conjectures are fairly standard and, if true, would represent the best possible range of p, q, r, modulo endpoints and interpolation with L^∞ estimates. This may be seen by testing the operators on characteristic functions of balls and k-plates L_δ. We also expect a suitable version of Conjecture 1.1 to hold without the requirement that f is supported on a fixed ball. However, by taking $f(x) = (1+|x|)^{-k}$, one sees that there is no hope for a global version of Conjecture 1.3.

All three conjectures, and even more, are known to hold when $k = d-1$, see [11], [30], and [21]. Conjectures 1.1 and 1.2 are known when $k > \frac{d}{2}$, or for any k when $p \leq \frac{d+1}{k+1}$, see [10]. In these cases there is no restriction to the local version of Conjecture 1.1.

Our results, Theorems 2.6 and 2.4, are interpolants of certain conjectured bounds. For example, from Conjecture 1.2 we expect (1.4) to hold with $p = \frac{d}{k} - \frac{d-k}{k2^k}$ and $q = d + \frac{k}{2^k-1}$. This implies the weaker estimate with $\tilde{p} = p$ and $\tilde{q} = 2p < q$. Interpolating this estimate with the trivial $L^\infty \to L^\infty$ estimate would yield Theorem 2.6.

1.2.2 Estimates for the X-ray transform

As discussed above, Proposition 2.5 motivates us to seek mixed-norm estimates for the X-ray transform T^1. From (1.3), we see that estimates for the X-ray transform imply bounds for the Kakeya maximal operator $\mathcal{M}^1_\delta$. This implication does not seem to be reversible, however it is reasonable to expect that methods used to prove bounds for $\mathcal{M}^1_\delta$ may extend to give estimates for T^1. This approach was used previously by Łaba Tao and Wolff in [44] and [23], where Wolff's hairbrush method of [43] was generalized. Unfortunately, neither of these bounds seems to be useful for the method of Section 1.2.1. In [44] a bound is only given for $d = 3$, and $\mathcal{M}^2$ is already well understood when

$d = 3$. In [23], a bound is given for $d > 3$ but only in terms of an L^p Sobolev space of strictly positive order. This represents the loss of a large enough negative power of δ that the use of the bound is not advantageous for our purposes.

Since [43] there has been much further study of $\mathcal{M}^1_\delta$, culminating in Katz and Tao's work [20] where it was shown that when $d \geq 6$

$$\|\mathcal{M}^1_\delta\|_{L^{\frac{4d+3}{4}}(G(d,1))} \lesssim \delta^{-\left(\frac{3(d-1)}{4d+3}+\epsilon\right)} \|f\|_{L^{\frac{4d+3}{7}}(\mathbb{R}^d)} \tag{1.5}$$

for $\epsilon > 0$ arbitrarily small. This bound was proven using a "sliced" estimate. Interestingly, Bourgain's recursive metric-entropy estimate is a sort of sliced X-ray transform estimate, see [33], and seems to be the first appearance of this technique.

We answer the question of whether (1.5) can be extended to a mixed-norm estimate affirmatively.

Theorem 3.1. *When $d \geq 6$ and $\epsilon > 0$,*

$$\|T^1[f]\|_{L^q(L^r)} \lesssim \|f\|_{L^p(\mathbb{R}^d)} \tag{1.6}$$

for f supported on a fixed ball, where $p = \frac{4d+3}{7}, q = \frac{4d+3}{4} - \epsilon$, and $r = \frac{4d+3}{3} - \epsilon$.

Except for endpoints, this attains the conjectured range of p and q for the ratio $\frac{r}{p} = \frac{7}{3}$. In contrast to [44] and [23], the refinements needed to obtain the mixed-norm estimate are quite minimal. An interesting feature of our proof is that we manage to avoid discretization arguments.

We also extend Katz and Tao's Hausdorff dimension estimate from [20] to obtain

Theorem 3.2. *For $d \geq 4$ and $\epsilon > 0$, there exist $p_\epsilon, q_\epsilon, r_\epsilon$ so that (1.6) holds for f supported on a fixed ball with $(p, q, r) = (p_\epsilon, q_\epsilon, r_\epsilon)$ where*

$$\frac{r_\epsilon}{p_\epsilon} > 1 + \sqrt{2} - \epsilon, \text{ and } \frac{q_\epsilon}{p_\epsilon} > 1 + \frac{\sqrt{2}}{2} - \epsilon.$$

This bound features an increased value of $\frac{r}{p}$, but not the sharp range of p for the given $\frac{r}{p}$. Using an observation [40] of Tao, we also obtain variants of Theorems 3.1 and 3.2 where the order of the mixed-norms is reversed. The correspondence between these "Nikodym-order" mixed-norms and the Nikodym maximal operator is analogous to that between the "Kakeya-order" mixed-norms (1.1) and the Kakeya maximal operator.

1.2.3 The Lebesgue measure and dimension of (d, k) sets

By combining Theorem 3.2 with the method of Section 1.2.1, we obtain

Theorem 1.4. *Suppose* $(1+\sqrt{2})^{k-1}+k > d$. *Then every* (d, k) *set has positive measure.*

Thus, for example, we now know that there are no $(8, 3)$ or $(18, 4)$ Besicovitch sets, where before we only knew that there were no $(7, 3)$ or $(10, 4)$ Besicovitch sets. We point out that the $(8, 3)$ estimate starts with a bound for $\mathcal{M}_\delta^1$ on $\mathbb{R}^6$, and the best known bound is Wolff's. By plugging this bound into Bourgain's original method, one may see that $(8, 3)$ sets have full Hausdorff dimension, but not positive measure. Thus, a mixed-norm estimate was necessary for the improvement.

We also obtain a bound for $\mathcal{M}^k$ corresponding to Theorem 1.4, but it is for extremely large p.

When $(1 + \sqrt{2})^{k-1} + k \leq d$, we do not know that (d, k) sets have positive measure, but we still have the following

Theorem 1.5. *The Hausdorff dimension of each* (d, k) *set is at least*

$$\max\left(d - \frac{d-k}{(1+\sqrt{2})^k}, \min\left(d, d+1-\frac{d-k}{(1+\sqrt{2})^{k-1}}\right)\right).$$

The estimates given by Theorems 1.4 and 1.5 are superior to those given by the direct use of the bush or hairbrush arguments such as [1] and [27]. In fact this would still be the case without the introduction of the mixed-norm estimate, if one were to update Bourgain's proof with Wolff's maximal operator bound. One may also attempt to directly apply arithmetic-combinatorial techniques to the (d, k) Kakeya problem, but the author's efforts in this regard have fallen short of Theorems 1.4 and 1.5.

Theorems 1.4 and 1.5 are proven in Chapter 2.

1.2.4 Finite fields

The Kakeya problem may also be considered in the setting of vector spaces over finite fields. In this discrete setting, one avoids certain technical issues present in Euclidean space, and some aspects of the arguments become more transparent.

Given a finite field F, a subset E of F^d is said to be a (d, k) set if it contains a translate of every k-dimensional subspace of F^d. By deriving analogues of Proposition 2.5 and Theorem 3.1, and using the analogue of Wolff's maximal operator bound from [29], we obtain the theorem below, where $|\cdot|$ denotes cardinality.

Theorem 4.2. *Suppose that the characteristic of F is strictly greater than 3. Then for every (d, k) set E in F^d*

$$|E| \gtrsim |F|^\eta$$

where

$$\eta = \max\left(d - \left(\frac{3}{7}\right)^k (d-k),\ d - \left(\frac{3}{7}\right)^{k-1} \frac{d-k-1}{2}\right).$$

and where the implicit constant is independent of $|F|$.

This is the finite field analogue of the statement that (d, k) sets have Minkowski dimension at least η.

Chapter 2

Maximal operators associated with k-planes

This chapter concerns the proof and application of Proposition 2.5, which should be considered a revisionist approach to Bourgain's recursive metric entropy estimate from [5].

As stated in Chapter 1, our main interest in Proposition 2.5 is the following principle. Given a bound

$$\|\mathcal{M}_\delta^{k-1}[f]\|_{L^{q_0}(G(d-1,k-1))} \lesssim \delta^{-\frac{\alpha_0}{p_0}} \|f\|_{L^{p_0}(\mathbb{R}^{d-1})}, \tag{2.1}$$

and a bound

$$\|T^1[f]\|_{L^q(L^r)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}, \tag{2.2}$$

we may obtain a bound

$$\|\mathcal{M}_\delta^{k}[f]\|_{L^{q_1}(G(d,k))} \lesssim \delta^{-\frac{\alpha_1}{p_1}} \|f\|_{L^{p_1}(\mathbb{R}^d)} \tag{2.3}$$

where $\alpha_1 = \alpha_0 \frac{p}{r}$. The value α_1 is of importance because the bound (2.3) implies that (d,k) sets have Hausdorff dimension at least $d - \alpha_1$. The specific statement of this principle is below.

Proposition 2.1. *Suppose the bound (2.1) holds for $f \in L^{p_0}(\mathbb{R}^{d-1})$ where $q_0 \geq p_0$, and suppose the bound (2.2) holds for $f \in L^p(\mathbb{R}^d)$ supported on a fixed ball where $q, r \geq p$.*

Then (2.3) holds for $f \in L^{p_1}(\mathbb{R}^d)$ where

$$q_1 = \min\left(q_0 \frac{r}{p_0}, q\right), \; p_1 = p$$

if $r \geq p_0$, and where

$$q_1 = \min\left(q_0, q\frac{p_0}{r}\right), \; p_1 = p\frac{p_0}{r}$$

if $r \leq p_0$.

Typically, this is applied with $r \geq p_0$. We point out that in either case $q_1 \geq p_1$, and so it is possible to use Proposition 2.1 recursively, provided bounds for T^1 are known in all dimensions with $q, r \geq p$. We delay the proofs of Propositions 2.1 and 2.5 until Section 2.1, where we also show that Theorem 2.6 follows from a combination of Drury and Christ's $L^{\frac{d+1}{2}} \to L^{d+1}(L^{d+1})$ bound for T^1 with Proposition 2.1.

One could formulate an endless number of variations of Theorem 2.6 by starting from a different bound for $\mathcal{M}^1_\delta$, using a different estimate for T^1, or considering the mixed-norm version of Proposition 2.5 from Section 2.2. One such variation is

Theorem 2.2. *Suppose $1 \leq k < d$. For every $\epsilon > 0$, there exists a $p < \infty$ so that*

$$\|\mathcal{M}^k_\delta[f]\|_{L^p(G(d,k))} \lesssim \delta^{-\frac{\alpha+\epsilon}{p}} \|f\|_{L^p(\mathbb{R}^d)}$$

where $\alpha = \frac{d-k}{(1+\sqrt{2})^k}$.

This follows immediately from Theorem 3.2 and Proposition 2.1.

The method above gives diminishing returns for small α. In particular, when $\alpha < 1 + \frac{1}{\sqrt{2}}$ it is preferable to use the L^2 method from [5] which we restate in a somewhat generalized form.

Proposition 2.3. *Suppose $k, p \geq 2$ and that a bound for $\mathcal{M}_\delta^{k-1}$ on $L^p(\mathbb{R}^{d-1})$ of the form*

$$\|\mathcal{M}_\delta^{k-1}[f]\|_{L^p(G(d-1,k-1))} \lesssim \delta^{-\frac{\alpha}{p}} \|f\|_{L^p(\mathbb{R}^{d-1})} \tag{2.4}$$

is known. Then if $\alpha \geq 1$ we have the bound

$$\|\mathcal{M}_\delta^{k}[f]\|_{L^p(G(d,k))} \lesssim \delta^{-\frac{\alpha-1}{p}} \|f\|_{L^p(\mathbb{R}^{d})} \tag{2.5}$$

for $f \in L^p(\mathbb{R}^d)$. If $\alpha < 1$ we have the bound

$$\|\mathcal{M}^{k}[f]\|_{L^p(G(d,k))} \lesssim \|f\|_{L^p(\mathbb{R}^{d})} \tag{2.6}$$

for $f \in L^p(\mathbb{R}^d)$ supported on a fixed ball.

This is proven in Section 2.3.

If $2^{k-1} + k > d$ and $k \geq 3$, we see from Theorem 2.6 that

$$\|\mathcal{M}_\delta^{k-2}[f]\|_{L^{\frac{d-1}{2}}(G(d-2,k-2))} \lesssim \delta^{-\frac{\alpha}{\frac{d-1}{2}}} \|f\|_{L^{\frac{d-1}{2}}(\mathbb{R}^{d-2})}$$

where $\alpha < 2$. Thus, applying (2.5) we see that

$$\|\mathcal{M}_\delta^{k-1}[f]\|_{L^{\frac{d-1}{2}}(G(d-1,k-1))} \lesssim \delta^{-\frac{\alpha'}{\frac{d-1}{2}}} \|f\|_{L^{\frac{d-1}{2}}(\mathbb{R}^{d-1})}$$

where $\alpha' < 1$. Applying (2.6), we obtain

Theorem 2.4. *Suppose $2^{k-1} + k > d$. Then*

$$\|\mathcal{M}^{k}[f]\|_{L^{\frac{d-1}{2}}(G(d,k))} \lesssim \|f\|_{L^{\frac{d-1}{2}}(\mathbb{R}^{d})}$$

for f supported on a fixed ball.

If $(1+\sqrt{2})^{k-1} + k > d$ and $k \geq 2$, then we see from Theorem 2.2 that for sufficiently large p

$$\|\mathcal{M}_\delta^{k-1}[f]\|_{L^p(G(d-1,k-1))} \lesssim \delta^{-\frac{\alpha}{p}} \|f\|_{L^p(\mathbb{R}^{d-1})}$$

where $\alpha < 1$. Thus (2.6) gives

$$\|\mathcal{M}^k[f]\|_{L^p(G(d,k))} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$$

for f supported on a fixed ball, which implies Theorem 1.4. Theorem 1.5 follows from Theorem 2.2 and a combination of Theorem 2.2 with (2.5).

2.1 A recursive maximal operator bound

We start with the definition of the measure we will use on $G(d,k)$. Fix any $L \in G(d,k)$. For a Borel subset F of $G(d,k)$ let

$$\mathcal{G}^{(d,k)}(F) = \mathcal{O}(\{\theta \in O(d) : \theta(L) \in F\})$$

where $\mathcal{O}$ is normalized Haar measure of the orthogonal group on $\mathbb{R}^d$, $O(d)$. By the transitivity of the action of $O(d)$ on $G(d,k)$ and the invariance of $\mathcal{O}$, it is clear that the definition is independent of the choice of L. Also note that $\mathcal{G}^{(d,k)}$ is invariant under the action of $O(d)$. By the uniqueness of uniformly-distributed measures (see [25], pages 44-53), $\mathcal{G}^{(d,k)}$ is the unique normalized Radon measure on $G(d,k)$ invariant under $O(d)$.

It will be necessary to use an alternate formulation of $\mathcal{G}^{(d,k)}$. For each ξ in $\mathbb{S}^{d-1}$ let $U_\xi : \xi^\perp \to \mathbb{R}^{d-1}$ be an orthogonal linear transformation. Then U_ξ^{-1} identifies $G(d-1,k-1)$ with the $k-1$ dimensional subspaces of $\xi^\perp$. Now, define $U : \mathbb{S}^{d-1} \times G(d-1,k-1) \to G(d,k)$ by

$$U(\xi, M) = \operatorname{span}(\xi, U_\xi^{-1}(M)).$$

Choosing U_ξ continuously on the upper and lower hemispheres of $\mathbb{S}^{d-1}$, U^{-1} identifies the Borel subsets of $G(d,k)$ with the completion of the Borel subsets of $\mathbb{S}^{d-1} \times G(d-

$1, k-1)$. Let σ^{d-1} denote normalized surface measure on the unit sphere. We claim that $\sigma^{d-1} \times \mathcal{G}^{(d-1,k-1)}(U^{-1}(F))$ is rotation invariant, and thus

$$\mathcal{G}^{(d,k)}(F) = \sigma^{d-1} \times \mathcal{G}^{(d-1,k-1)}(U^{-1}(F)). \tag{2.7}$$

Indeed, let $R \in O(d)$. Then for $F \subset G(d,k)$

$$\begin{aligned}\int_{\mathbb{S}^{d-1}} \int_{G(d-1,k-1)} \chi_{RF}(U(\xi, M))\, dM\, d\xi \\ &= \int_{\mathbb{S}^{d-1}} \int_{G(d-1,k-1)} \chi_F(\mathrm{span}(R^{-1}\xi, R^{-1}U_\xi^{-1}(M)))\, dM\, d\xi \\ &= \int_{\mathbb{S}^{d-1}} \int_{G(d-1,k-1)} \chi_F(\mathrm{span}(R^{-1}\xi, U_{R^{-1}\xi}^{-1}(R_\xi M)))\, dM\, d\xi \\ &= \int_{\mathbb{S}^{d-1}} \int_{G(d-1,k-1)} \chi_F(\mathrm{span}(R^{-1}\xi, U_{R^{-1}\xi}^{-1}(M)))\, dM\, d\xi \\ &= \int_{\mathbb{S}^{d-1}} \int_{G(d-1,k-1)} \chi_F(\mathrm{span}(\xi, U_\xi^{-1}(M)))\, dM\, d\xi.\end{aligned}$$

Above, each R_ξ is a suitable element of $O(d-1)$. The second equation follows from the orthogonality of U_ξ and R, and the appropriate choice of R_ξ. The third equation follows from the fact that $\mathcal{G}^{(d-1,k-1)}$ is invariant under $O(d-1)$. The last equation follows from the fact that σ^{d-1} is invariant under $O(d)$.

Proposition 2.5. *Suppose the bounds*

$$\|\mathcal{M}_\delta^{k-1}[f]\|_{L^{q_0}(G(d-1,k-1))} \lesssim \delta^{-\frac{\alpha_0}{p_0}} \|f\|_{L^{p_0}(\mathbb{R}^{d-1})} \tag{2.8}$$

and

$$\|T^1[g]\|_{L^{q_0}(L^{p_0})} \lesssim \|g\|_{L^{p_1}(\mathbb{R}^d)}$$

are known to hold for $f \in L^{p_0}(\mathbb{R}^{d-1})$ and for $g \in L^{p_1}(\mathbb{R}^d)$ supported on a fixed ball, where $q_0 \geq p_1$. Then the bound

$$\|\mathcal{M}_\delta^{k}[f]\|_{L^{q_0}(G(d,k))} \lesssim \delta^{-\frac{\alpha_1}{p_1}} \|f\|_{L^{p_1}(\mathbb{R}^d)}$$

holds for $f \in L^{p_1}(\mathbb{R}^d)$, where $\alpha_1 = \alpha_0 \frac{p_1}{p_0}$.

Proof. Without loss of generality, we assume that f is positive. Since averaging over a k-plate is local and we are proving an $L^p \to L^q(L^r)$ estimate with $p \leq q \leq r$, we may also assume that f is supported on a fixed ball.

Let $L \in G(d,k)$ and suppose that $L = \operatorname{span}(\xi, U_\xi^{-1}(M))$ where $M \in G(d-1,k-1)$. Let $a_L \in \mathbb{R}^d$ and let $a_M = U_\xi(\operatorname{proj}_{\xi^\perp}(a_L))$, where proj denotes orthogonal projection. Then

$$\begin{aligned}\int_{L_\delta(a_L)} f(y)\ dy &\leq \int_{M_\delta(a_M)} \int_{\mathbb{R}} f(U_\xi^{-1}(x) + t\xi)\ dt\ dx \\ &= \int_{M_\delta(a_M)} T^1[f](\xi, U_\xi^{-1}(x))\ dx\end{aligned}$$

where $L_\delta(a_L)$ and $M_\delta(a_M)$ are k and $k-1$ plates respectively. Noting that $d-k = (d-1)-(k-1)$, it follows that

$$\mathcal{M}_\delta^k[f](L) \lesssim \mathcal{M}_\delta^{k-1}[T^1[f](\xi, U_\xi^{-1}(\cdot))](M).$$

By (2.7) and our hypothesized bound for $\mathcal{M}_\delta^{k-1}$, we now have

$$\begin{aligned}\|\mathcal{M}_\delta^k[f]\|_{L^{q_0}(G(d,k))} &\lesssim \left(\int_{\mathbb{S}^{d-1}} \int_{G(d-1,k-1)} \mathcal{M}_\delta^{k-1}[T^1[f](\xi, U_\xi^{-1}(\cdot))](M)^{q_0}\ dM\ d\xi\right)^{\frac{1}{q_0}} \\ &\lesssim \delta^{-\frac{\alpha_0}{p_0}} \left(\int_{\mathbb{S}^{d-1}} \left(\int_{\mathbb{R}^{d-1}} T^1[f](\xi, U_\xi^{-1}(x))^{p_0}\ dx\right)^{\frac{q_0}{p_0}} d\xi\right)^{\frac{1}{q_0}} \\ &= \delta^{-\frac{\alpha_0}{p_0}} \|T^1[f]\|_{L^{q_0}(L^{p_0})}.\end{aligned}$$

Finally, f is supported on a fixed ball so we may apply our hypothesized estimate for T^1

$$\delta^{-\frac{\alpha_0}{p_0}} \|T^1[f]\|_{L^{q_0}(L^{p_0})} \lesssim \delta^{-\frac{\alpha_0}{p_0}} \|f\|_{L^{p_1}(\mathbb{R}^d)} = \delta^{-\frac{\alpha_1}{p_1}} \|f\|_{L^{p_1}(\mathbb{R}^d)}.$$

□

Proof of Proposition 2.1. First, suppose $r \geq p_0$. Then, we may interpolate our hypothesized bound for $\mathcal{M}_\delta^{k-1}$ with the trivial L^∞ bound to obtain the bound

$$\|\mathcal{M}_\delta^{k-1}[f]\|_{L^{q_0 \frac{r}{p_0}}(G(d-1,k-1))} \lesssim \delta^{-\frac{\alpha_0}{r}} \|f\|_{L^r(\mathbb{R}^{d-1})}.$$

Applying Hölder's inequality if necessary, the same bound holds with $q_0 \frac{r}{p_0}$ replaced by q_1. Possibly applying Hölder's inequality again, we obtain from our hypothesized estimate for T^1,

$$\|T^1[f]\|_{L^{q_1}(L^r)} \lesssim \|f\|_{L^{p_1}(\mathbb{R}^d)}.$$

By our assumptions that $q_o \geq p_0$ and $q, r \geq p$, we have $q_1 \geq p_1$ and so we may apply Proposition 2.5 to obtain the conclusion.

Now, suppose $r \leq p_0$. Since we are only considering bounds for $T^1[f]$ when f is supported on a fixed ball, we may interpolate with the trivial $L^\infty \to L^\infty(L^\infty)$ bound to obtain

$$\|T^1[f]\|_{L^{q \frac{p_0}{r}}(L^{p_0})} \lesssim \|f\|_{L^{p_1}(\mathbb{R}^d)}$$

where $\frac{1}{p_1} = \frac{r}{p_0}\frac{1}{p}$. Applying Hölder's inequality if necessary, we may replace $q\frac{p_0}{r}$ by q_1. Possibly applying Hölder's inequality again, we obtain the bound

$$\|\mathcal{M}_\delta^{k-1}[f]\|_{L^{q_1}(G(d-1,k-1))} \lesssim \delta^{-\frac{\alpha_0}{p_0}} \|f\|_{L^{p_0}(\mathbb{R}^{d-1})}.$$

By our assumptions that $q_0 \geq p_0$ and $q, r \geq p$, we have $q_1 \geq p_1$ and so we may apply Proposition 2.5 to obtain the conclusion. □

Theorem 2.6. *For $1 \leq k < d$*

$$\|\mathcal{M}_\delta^k[f]\|_{L^q(G(d,k))} \lesssim \delta^{-\frac{\alpha}{p}} \|f\|_{L^p(\mathbb{R}^d)}$$

where $p = \frac{d+1}{2}$, $q = d+1$, and $\alpha = \frac{d-k}{2^k}$.

Proof. We first observe the well known fact that Theorem 2.6 holds when $k = 1$. From Hölder's inequality, we see that for any $L \in G(d, 1)$ and $a \in \mathbb{R}^d$

$$\int_{L_\delta(a)} |f(x)|\, dx \lesssim \delta^{(d-1)(1-\frac{1}{d+1})} \left(\int_{\mathbb{R}^{d-1}} |T^1[f](L, x)|^{d+1}\, dx \right)^{\frac{1}{d+1}}.$$

Thus, using the $L^{\frac{d+1}{2}} \to L^{d+1}(L^{d+1})$ bound from [10] for the second inequality,

$$\begin{aligned} \|\mathcal{M}^1_\delta[f]\|_{L^{d+1}(G(d,1))} &\lesssim \delta^{-(d-1)} \delta^{(d-1)(1-\frac{1}{d+1})} \|T^1[f]\|_{L^{d+1}(L^{d+1})} \\ &\lesssim \delta^{-(d-1)} \delta^{(d-1)(1-\frac{1}{d+1})} \|f\|_{L^{\frac{d+1}{2}}(\mathbb{R}^d)} \\ &= \delta^{-\frac{d-1}{2}/\frac{d+1}{2}} \|f\|_{L^{\frac{d+1}{2}}(\mathbb{R}^d)}. \end{aligned}$$

Now, assume that Theorem 2.6 holds for $(d-1, k-1)$. Then, we may take $\alpha_0 = \frac{d-1-(k-1)}{2^{k-1}} = \frac{d-k}{2^{k-1}}$, $p_0 = \frac{d}{2}$, and $q_0 = d$ in Proposition 2.1. From the $L^{\frac{d+1}{2}} \to L^{d+1}(L^{d+1})$ bound for T^1, we take $p = \frac{d+1}{2}$, $q = d+1$, and $r = d+1$. Thus $q_1 = \min(d\frac{d+1}{\frac{d}{2}}, d+1) = d+1$, $p_1 = \frac{d+1}{2}$, and $\alpha_1 = \frac{d-k}{2^{k-1}}\frac{1}{2} = \frac{d-k}{2^k}$. Hence, Theorem 2.6 holds for (d, k). $\square$

2.2 A recursive mixed-norm estimate

We now demonstrate that the method of the previous section can also be used to obtain mixed-norm estimates for T^k. Our recursive bound here is

Proposition 2.7. *Suppose the bounds*

$$\|T^{k-1}[f]\|_{L^{q_0}(L^{r_0})} \lesssim \|f\|_{L^{p_0}(\mathbb{R}^{d-1})} \tag{2.9}$$

and

$$\|T^1[g]\|_{L^{q_0}(L^{p_0})} \lesssim \|g\|_{L^{p_1}(\mathbb{R}^d)}$$

are known to hold for $f \in L^{p_0}(\mathbb{R}^{d-1})$ *supported on a fixed ball, and for* $g \in L^{p_1}(\mathbb{R}^d)$ *supported on a fixed ball. Then the bound*

$$\|T^k[f]\|_{L^{q_0}(L^{r_0})} \lesssim \|f\|_{L^{p_1}(\mathbb{R}^d)}$$

holds for $f \in L^{p_1}(\mathbb{R}^d)$ *supported on a fixed ball.*

Instead of reducing the quantity α, we apply Proposition 2.7 to increase the ratio $\frac{r_0}{p_0}$. Given a bound

$$\|T^{k-1}[f]\|_{L^{q_0}(L^{r_0})} \lesssim \|f\|_{L^{p_0}(\mathbb{R}^{d-1})}, \tag{2.10}$$

and a bound

$$\|T^1[f]\|_{L^q(L^r)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}, \tag{2.11}$$

we may obtain a bound

$$\|T^k[f]\|_{L^{q_1}(L^{r_1})} \lesssim \|f\|_{L^{p_1}(\mathbb{R}^d)} \tag{2.12}$$

where $\frac{r_1}{p_1} = \frac{r_0}{p_0}\frac{r}{p}$. The specific statement of this principle is below. We will not give any applications; let it suffice to say that one may obtain previously unknown estimates, but these estimates are rarely for the optimal range of p and never for the optimal range of q (here we mean the optimal range for the given value of $\frac{r}{p}$).

Proposition 2.8. *Suppose the bound (2.10) holds for* $f \in L^{p_0}(\mathbb{R}^{d-1})$ *supported on a fixed ball, and suppose the bound (2.11) holds for* $f \in L^p(\mathbb{R}^d)$ *supported on a fixed ball. Then (2.12) holds for* $f \in L^{p_1}(\mathbb{R}^d)$ *supported on a fixed ball, where*

$$q_1 = \min\left(q_0\frac{r}{p_0}, q\right),\ r_1 = r_0\frac{r}{p_0},\ p_1 = p$$

if $r \geq p_0$*, and where*

$$q_1 = \min\left(q_0, q\frac{p_0}{r}\right),\ r_1 = r_0,\ p_1 = p\frac{p_0}{r}$$

if $r \leq p_0$.

Proof of Proposition 2.7. Without loss of generality, we assume that f is positive.

Let $L \in G(d,k)$, $x \in L^\perp$, and suppose that $L = \operatorname{span}(\xi, U_\xi^{-1}(M))$ where $M \in G(d-1,k-1)$. Then

$$T^k[f](L,x) = \int_{L+x} f(y)\ dy = \int_{M+U_\xi(x)} \int_{\mathbb{R}} f(U_\xi^{-1}(z) + t\xi)\ dt\ dz$$
$$= T^{k-1}[T^1[f](\xi, U_\xi^{-1}(\cdot))](M, U_\xi(x)).$$

By (2.7) and our hypothesized bound for T^{k-1}, we now have

$$\begin{aligned}
&\|T^k[f]\|_{L^{q_0}(L^{r_0})} \\
&\lesssim \left(\int_{\mathbb{S}^{d-1}} \int_{G(d-1,k-1)} \left(\int_{M^\perp} T^{k-1}[T^1[f](\xi, U_\xi^{-1}(\cdot))](M,x)^{r_0}\ dx\right)^{\frac{q_0}{r_0}}\ dM\ d\xi\right)^{\frac{1}{q_0}} \\
&\lesssim \left(\int_{\mathbb{S}^{d-1}} \left(\int_{\mathbb{R}^{d-1}} T^1[f](\xi, U_\xi^{-1}(y))^{p_0}\ dy\right)^{\frac{q_0}{p_0}} d\xi\right)^{\frac{1}{q_0}} \\
&= \|T^1[f]\|_{L^{q_0}(L^{p_0})},
\end{aligned}$$

where above we note that $T^1[f](\xi, U_\xi^{-1}(\cdot))$ is supported on a fixed ball, since f is supported on a fixed ball.

Finally, we apply apply our hypothesized bound for T^1

$$\|T^1[f]\|_{L^{q_0}(L^{p_0})} \lesssim \|f\|_{L^{p_1}(\mathbb{R}^d)}.$$

□

Proof of Proposition 2.8. First, suppose $r \geq p_0$. Then, since we are only considering local estimates, we may interpolate our known bound for T^{k-1} with the trivial L^∞

bound to obtain the bound

$$\|T^{k-1}[f]\|_{L^{q_0 \frac{r}{p_0}}\left(L^{r_0 \frac{r}{p_0}}\right)} \lesssim \|f\|_{L^r(\mathbb{R}^{d-1})}.$$

Applying Hölder's inequality if necessary, the same bound holds with $q_0 \frac{r}{p_0}$ replaced by q_1. Possibly applying Hölder's inequality again, we obtain from our hypothesized estimate for T^1,

$$\|T^1[f]\|_{L^{q_1}(L^r)} \lesssim \|f\|_{L^{p_1}(\mathbb{R}^d)}.$$

We apply Proposition 2.5 to obtain the conclusion.

Now, suppose $r \leq p_0$. Since we are only considering local estimates, we may interpolate with the trivial L^∞ bound to obtain

$$\|T^1[f]\|_{L^{q\frac{p_0}{r}}(L^{p_0})} \lesssim \|f\|_{L^{p_1}(\mathbb{R}^d)}.$$

Applying Hölder's inequality if necessary, we may replace $q\frac{p_0}{r}$ by q_1. Possibly applying Hölder's inequality again, we obtain the bound

$$\|T^{k-1}[f]\|_{L^{q_1}(L^{r_0})} \lesssim \|f\|_{L^{p_0}(\mathbb{R}^{d-1})}.$$

We apply Proposition 2.5 to obtain the conclusion. □

2.3 The L^2 method

Reducing α_0 by a fixed factor $\frac{r}{p}$, as in Proposition 2.1, is not a substantial gain for small α_0. By using an L^2 estimate of the X-ray transform which takes advantage of cancellation, instead of the $L^p \to L^q(L^r)$ bounds, we may take $\alpha_1 = \alpha_0 - 1$ when $\alpha_0 \geq 1$ and obtain a bound for $\mathcal{M}^k$ when $\alpha_0 < 1$.

The main estimate needed to derive Proposition 2.3 was proven by Smith and Solmon in [37].

Lemma 2.9. *For $d \geq 3$*

$$\|T^1[f]\|_{L^2(L^2)} = C_d \|f\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)}$$

where C_d is a fixed constant depending only on d and $\dot{H}$ denotes the homogeneous L^2 Sobolev space.

It immediately follows that if the Fourier transform $\widehat{g}$ of a function g is identically 0 on $B(0,R)$ then

$$\|T^1[g]\|_{L^2(L^2)} \lesssim R^{-\frac{1}{2}} \|g\|_{L^2(\mathbb{R}^d)}. \tag{2.13}$$

To effectively apply (2.13), we use a Littlewood-Paley decomposition. Let ϕ_0 be a Schwartz function with $\widehat{\phi_0} \equiv 1$ on $B(0,1)$ and with $\widehat{\phi_0}$ supported on $B(0,2)$. For $j > 0$, define $\phi_j = 2^{jd}\phi_0(2^j \cdot) - 2^{(j-1)d}\phi_0(2^{j-1}\cdot)$ so that $\widehat{\phi_j}$ is supported on $B(0,2^{j+1}) \setminus B(0,2^{j-1})$. Functions are decomposed

$$f = \sum_{j=0}^{\infty} f_j$$

where $f_j = f * \phi_j$.

Our last ingredients are two Schwartz-tail estimates needed to reconcile the localization properties of the space and frequency variables.

Lemma 2.10. *Suppose $g \geq 0$ and $\widehat{\tilde{g}} = \widehat{g}$ on $B(0, \frac{1}{\delta})$. Then*

$$\mathcal{M}_\delta^{k-1}[g] \lesssim \mathcal{M}_\delta^{k-1}[|\tilde{g}|]. \tag{2.14}$$

Proof. For $1 \leq n \leq d$ let Φ^n be a nonnegative Schwartz function on $\mathbb{R}^n$ such that $\Phi^n \geq 1$ on $B(0,1)$ and $\widehat{\Phi}^n$ is supported on $B\left(0, \frac{1}{\sqrt{2}}\right)$. For $L \in G(d,k)$ let

$$\pi_{L,\delta}(x) = \Phi^k(\mathrm{proj}_L(x))\delta^{-(d-k)}\Phi^{d-k}\left(\mathrm{proj}_{L^\perp}\left(\frac{x}{\delta}\right)\right).$$

Now, define

$$\widetilde{\mathcal{M}}_\delta^k[f](L) = \sup_{a\in\mathbb{R}^d} \int_{\mathbb{R}^d} \pi_{L,\delta}(x-a)f(x)dx.$$

By construction, $\pi_{L,\delta}(\cdot - a) \gtrsim \frac{\chi_{L_\delta(a)}}{\mathcal{L}^d(L_\delta(a))}$ and $\widehat{\pi}_{L,\delta}$ is supported on $B(0,\frac{1}{\delta})$. Thus

$$\mathcal{M}_\delta^k[g] \lesssim \widetilde{\mathcal{M}}_\delta^k[g] = \widetilde{\mathcal{M}}_\delta^k[\tilde{g}].$$

Since Φ^k and Φ^{d-k} are Schwartz functions, we have

$$\Phi^k \leq \sum_{j=1}^{\infty} c_j \chi_{B(y_j,\frac{1}{2})}$$

and

$$\Phi^{d-k} \leq \sum_{j=1}^{\infty} d_j \chi_{B(z_j,\frac{1}{2})}$$

for some $\{c_j\},\{d_j\} \in l^1(\mathbb{N})$, $\{y_j\} \subset \mathbb{R}^k$, and $\{z_j\} \subset \mathbb{R}^{d-k}$. Then, for an appropriately chosen $\{a_{j,l}\}$

$$\begin{aligned}\pi_{L,\delta}(x) &\leq \sum_{j,l=1}^{\infty} c_j d_l \chi_{B(y_j,\frac{1}{2})}(\mathrm{proj}_L(x))\delta^{-(d-k)}\chi_{B(z_l,\frac{1}{2})}\left(\mathrm{proj}_{L^\perp}\left(\frac{x}{\delta}\right)\right)\\ &\lesssim \sum_{j,l=1}^{\infty} c_j d_l \frac{\chi_{L_\delta(a_{j,l})}}{\mathcal{L}^d(L_\delta(a_{j,l}))}.\end{aligned}$$

Thus,

$$\widetilde{\mathcal{M}}_\delta^k[\tilde{g}] \lesssim \sum_{j,l=1}^{\infty} c_j d_l \mathcal{M}_\delta^k[|\tilde{g}|] \lesssim \mathcal{M}_\delta^k[|\tilde{g}|].$$

□

Lemma 2.11. *Define*

$$\mathcal{M}_{\mathrm{loc}}^{k-1}[g](L) = \sup_{a\in\mathbb{R}^d} \int_{a+(L\cap B(0,\frac{1}{2}))} g(x)dx.$$

Suppose $\widehat{g}$ *is supported on* $B(0,\frac{1}{\delta})$. *Then*

$$\mathcal{M}_{\mathrm{loc}}^{k-1}[|g|] \lesssim \mathcal{M}_\delta^{k-1}[|g|].$$

Proof. Since $\widehat{g}$ is supported on $B(0, \frac{1}{\delta})$,

$$g = g * \delta^{-d}\phi_0\left(\frac{\cdot}{\delta}\right)$$

and so

$$\int_{a+(L\cap B(0,\frac{1}{2})} |g(x)|dx \leq \int_{\mathbb{R}^d} \left|\delta^{-d}\phi_0\left(\frac{y}{\delta}\right)\right| \int_{a+y+(L\cap B(0,\frac{1}{2}))} |g(x)|dx\ dy.$$

Since ϕ_0 is a Schwartz function,

$$|\phi_0| \leq \sum_{j=1}^{\infty} c_j \chi_{B(y_j,\frac{1}{2})}$$

for some $\{c_j\} \in l^1(\mathbb{N})$ and $\{y_j\} \subset \mathbb{R}^d$. Thus

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left|\delta^{-d}\phi_0\left(\frac{y}{\delta}\right)\right| \int_{a+y+(L\cap B(0,\frac{1}{2}))} |g(x)|dx\ dy \\
&\leq \sum_{j=1}^{\infty} c_j \delta^{-d} \int_{B(\delta y_j,\frac{\delta}{2})} \int_{a+y+(L\cap B(0,\frac{1}{2}))} |g(x)|dx\ dy \\
&\lesssim \sum_{j=1}^{\infty} c_j \mathcal{M}_\delta^{k-1}[|g|](L) \\
&\lesssim \mathcal{M}_\delta^{k-1}[|g|](L).
\end{aligned}$$

□

Proof of Proposition 2.3. We begin by proving (2.5). Averaging over each k-plate is local and we are proving an $L^p \to L^q(L^r)$ bound where $p \leq q \leq r$, so we may assume that f is supported on a fixed ball. Additionally, assume that f is nonnegative.

Following the proof of Proposition 2.5, we observe that for $L = \text{span}(\xi, U_\xi^{-1}(M))$ we have

$$\mathcal{M}_\delta^k[f](L) \lesssim \mathcal{M}_\delta^{k-1}[T^1[f](\xi, U_\xi^{-1}(\cdot))](M). \tag{2.15}$$

Since f is supported on a fixed ball, we may switch the order of integration between convolution and the X-ray transform to obtain

$$\|T^1[f_j]\|_{L^\infty(L^\infty)} \lesssim \|f\|_{L^\infty(\mathbb{R}^d)}$$

uniformly in j. Hence, interpolation with (2.13) gives

$$\|T^1[f_j]\|_{L^p(L^p)} \lesssim (2^{-j})^{\frac{1}{p}} \|f\|_{L^p(\mathbb{R}^d)} \tag{2.16}$$

for any $p \geq 2$.

From Lemma 2.10, we obtain

$$\mathcal{M}_\delta^{k-1}[T^1[f](\xi, U_\xi^{-1}(\cdot))](M) \lesssim \sum_{j=0}^{|\log(\delta)|+1} \mathcal{M}_\delta^{k-1}[|T^1[f_j](\xi, U_\xi^{-1}(\cdot))|](M). \tag{2.17}$$

Averaging Lemma 2.11 gives, for each j,

$$\mathcal{M}_\delta^{k-1}[|T^1[f_j](\xi, U_\xi^{-1}(\cdot))|](M) \lesssim \mathcal{M}_{2^{-j}}^{k-1}[|T^1[f_j](\xi, U_\xi^{-1}(\cdot))|](M). \tag{2.18}$$

Integrating over $G(d,k)$ and combining the bounds (2.4) and (2.16) as in the proof of Proposition 2.5, we obtain

$$\|\mathcal{M}_\delta^k[f]\|_{L^p(G(d,k))} \lesssim \sum_{j=0}^{|\log\delta|+1} (2^j)^{\frac{\alpha-1}{p}} \|f\|_{L^p(\mathbb{R}^d)} \lesssim \delta^{-\frac{\alpha-1}{p}} \|f\|_{L^p(\mathbb{R}^d)}$$

from (2.15), (2.17), and (2.18), when $\alpha \geq 1$.

Proceeding to the proof of (2.6), we have f supported on the unit ball and we assume that f is nonnegative, giving

$$\mathcal{M}^k[f] \lesssim \mathcal{M}_{\text{loc}}^k[f].$$

As before,

$$\mathcal{M}_{\text{loc}}^k[f](L) \lesssim \mathcal{M}_{\text{loc}}^{k-1}[T^1[f](\xi, U_\xi^{-1}(\cdot))](M)$$

and

$$\mathcal{M}^{k-1}_{\text{loc}}[|T^1[f_j](\xi, U_\xi^{-1}(\cdot))|](M) \lesssim \mathcal{M}^{k-1}_{2^{-j}}[|T^1[f_j](\xi, U_\xi^{-1}(\cdot))|](M),$$

giving

$$\|\mathcal{M}^k[f]\|_{L^p(G(d,k))} \lesssim \sum_{j=0}^{\infty}(2^j)^{\frac{\alpha-1}{p}}\|f\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$$

when $\alpha < 1$. □

Chapter 3

Estimates for the X-ray transform

In this chapter, we extend two results of Katz and Tao from [20] to give mixed-norm estimates. In addition to their intrinsic interest, these mixed-norm estimates are needed to fully utilize the method of Chapter 2. Modulo endpoints, Katz and Tao gave the best possible $L^{\frac{4d+3}{7}}$ bounds for $\mathcal{M}^1_\delta$ on $\mathbb{R}^d$, $d \geq 6$. Through a slight refinement of their technique, we give the best possible, except for endpoints, local $L^{\frac{4d+3}{7}}$ bounds for T^1 on $\mathbb{R}^d$, $d \geq 6$.

Theorem 3.1. *When $d \geq 6$ and $\epsilon > 0$,*

$$\|T^1[f]\|_{L^q(L^r)} \lesssim \|f\|_{L^p(\mathbb{R}^d)} \tag{3.1}$$

for f supported on a fixed ball, where $p = \frac{4d+3}{7}, q = \frac{4d+3}{4} - \epsilon$, and $r = \frac{4d+3}{3} - \epsilon$.

By combining an iterated version of the main estimate from their maximal operator bound with certain "hairbrush"-type improvements, Katz and Tao showed that $(d, 1)$ sets have Hausdorff dimension at least $(2 - \sqrt{2})(d - 4) + 3$. Ignoring the hairbrush improvements, which are perhaps not well suited for mixed-norm estimates, one would obtain the lower bound $(2 - \sqrt{2})(d - 4) + 3 - (3\sqrt{2} - 4)$. We extend this lower bound to a mixed-norm estimate for T^1.

Theorem 3.2. *For $d \geq 4$ and $\epsilon > 0$, there exist $p_\epsilon, q_\epsilon, r_\epsilon$ so that (3.1) holds for f*

supported on a fixed ball with $(p,q,r) = (p_\epsilon, q_\epsilon, r_\epsilon)$ *where*

$$\frac{r_\epsilon}{p_\epsilon} > 1 + \sqrt{2} - \epsilon, \ \textit{and} \ \frac{q_\epsilon}{p_\epsilon} > 1 + \frac{\sqrt{2}}{2} - \epsilon.$$

Here, we do not obtain the optimal range of p for the given ratio $\frac{r}{p}$. This is because, at present, we are not able to effectively use the two-ends reduction for iterated estimates.

It will convenient for our purposes to use a different parametrization of the set of lines. Let $e_1, \ldots, e_d$ be an orthonormal basis for $\mathbb{R}^d$ and let $H = \text{span}(e_1, \ldots, e_{d-1})$. For $\xi, x \in H$ and a function f defined on $\mathbb{R}^d$, the local X-ray transform of f at the line $x + \mathbb{R}(\xi + e_d)$ may be defined

$$T[f](\xi, x) = \int_0^1 f(x + t(\xi + e_d))\, dt.$$

We consider the "Kakeya-order" mixed norm of $T[f]$

$$\|T[f]\|_{L^q(L^r),K} = \left(\int_{B(0,C)} \left(\int_{\mathbb{R}^{d-1}} |T[f](\xi, x)|^r \, dx \right)^{\frac{q}{r}} \, d\xi \right)^{\frac{1}{q}} \tag{3.2}$$

where $C > 0$ and $B(0, C)$ denotes the ball centered at 0 with radius C in $\mathbb{R}^{d-1}$, and we aim to prove bounds of the form

$$\|T[f]\|_{L^q(L^r),K} \lesssim \|f\|_{L^p(\mathbb{R}^d)}. \tag{3.3}$$

Through a covering argument, one may see that the bound (3.3) is equivalent to the more conventional version

$$\|T^1[f]\|_{L^q(L^r)} \lesssim \|f\|_{L^p(\mathbb{R}^d)} \tag{3.4}$$

where we only consider f supported on a fixed ball, and where T^1 and the mixed-norms are as defined in Chapter 1.

By testing T on the characteristic functions of δ-neighborhoods of points and line segments, and letting δ approach 0, one sees that

$$1 + \frac{d-1}{r} \geq \frac{d}{p} \tag{3.5}$$

and

$$\frac{1}{q} + \frac{1}{r} \geq \frac{1}{p} \tag{3.6}$$

are necessary conditions for the bound (3.3) to hold. It is conjectured that, together with the condition $r < \infty$, these are also sufficient. This was shown to be the case for $p < \frac{d+1}{2}$ by Drury in [13] and for $p = \frac{d+1}{2}$ by Christ in [10].

Instead of the Kayeya-order mixed norms (3.2), one may also consider the Nikodym-order mixed norms of $T[f]$

$$\|T[f]\|_{L^q(L^r),N} = \left(\int_{\mathbb{R}^{d-1}} \left(\int_{B(0,C)} |T[f](\xi, x)|^r \, d\xi \right)^{\frac{q}{r}} \, dx \right)^{\frac{1}{q}}.$$

In order for the bound

$$\|T[f]\|_{L^q(L^r),N} \lesssim \|f\|_{L^p(\mathbb{R}^d)} \tag{3.7}$$

to hold, we again have the necessary conditions (3.5), (3.6), and $r < \infty$. Unless we impose the additional assumption that f is supported away from H, we have another necessary condition

$$1 + \frac{d-1}{q} \geq \frac{d}{p} \tag{3.8}$$

which follows from the application of T to characteristic functions of δ-neighborhoods of a point in H. Tao showed in [40] that bounds for the Kakeya and Nikodym maximal operators are roughly equivalent. We observe that his proof carries over to the general mixed norm case, and hence combined with Theorem 3.1 yields

Corollary 3.3. *When $d \geq 6$ and $\epsilon > 0$, the bound (3.7) holds with $p = \frac{4d+3}{7}, q = \frac{4d+3}{4} - \epsilon$, and $r = \frac{4d+3}{3} - \epsilon$.*

One may also formulate a corresponding version of Theorem 3.2.

We prove Theorem 3.1 in sections 3.1, 3.2, 3.3, and 3.4. We give the additional arguments needed for Theorem 3.2 in section 3.5. We show that Corollary 3.3 follows from Theorem 3.1 in section 3.6.

3.1 Reduction to weak estimates

We first note that, since T is local, when $p \leq q \leq r$ it suffices to prove (3.3) for f supported on the cube Q centered at $\frac{1}{2}e_d$ with side length 1. A natural simplification of the bound (3.3) is the estimate

$$\|\lambda \chi_F\|_{L^q(L^r),K} \lesssim \|\chi_E\|_{L^p(\mathbb{R}^d)} \tag{3.9}$$

where $E \subset Q$, $0 < \lambda \leq 1$, and

$$T[\chi_E](\xi, x) \geq \lambda \text{ for } (\xi, x) \in F \subset B(0, C) \times \mathbb{R}^{d-1}. \tag{3.10}$$

We further simplify the estimate (3.9) to

$$\begin{aligned} \mathcal{L}^d(E)^{\frac{1}{p}} &\gtrsim \lambda \left(\frac{\mathcal{L}^{2(d-1)}(F)}{\Omega(F)} \right)^{\frac{1}{q}} \Omega(F)^{\frac{1}{r}} \\ &= \lambda \mathcal{L}^{2(d-1)}(F)^{\frac{1}{q}} \Omega(F)^{\frac{1}{r} - \frac{1}{q}} \end{aligned} \tag{3.11}$$

where

$$\Omega(F) = \sup_{\xi \in B(0,C)} \mathcal{L}^{d-1}(\{x : (\xi, x) \in F\}).$$

The crude interpolation argument below shows that (3.3) follows from (3.11).

Claim 3.4. *Suppose that $r > q$ and that the estimate (3.11) holds for all E contained in Q. Then for any $\epsilon > 0$, the bound*

$$\|T[f]\|_{L^q(L^r),K} \lesssim \|f\|_{L^{p+\epsilon}(\mathbb{R}^d)}$$

holds for functions f supported on Q.

Proof. Without loss of generality, assume that f is nonnegative and

$$\|f\|_{L^{p+\epsilon}(\mathbb{R}^d)} = 1. \tag{3.12}$$

For integers j, k, l let

$$E_j = \{x \in \mathbb{R}^d : 2^{j-1} < f(x) \leq 2^j\}$$

$$F_{j,k} = \{(\xi, x) \in B(0, C) \times \mathbb{R}^{d-1} : 2^{k-1} < T[\chi_{E_j}](\xi, x) \leq 2^k\}$$

$$F_{j,k,l} = \{(\xi, x) \in F_{j,k} : 2^{l-1} < \mathcal{L}^{d-1}(\{x' : (\xi, x') \in F_{j,k}\}) \leq 2^l\}.$$

Since each $T[\chi_{E_j}] \leq 1$, we have $F_{j,k} = \emptyset$ for $k > 1$. Since we only consider $\xi \in B(0, C)$ and f supported on Q, $F_{j,k,l} = \emptyset$ for $l > \tilde{C}$. It follows that

$$\|T[f]\|_{L^q(L^r),K} \lesssim \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{0} \sum_{l=-\infty}^{\tilde{C}} 2^{j+k+l(\frac{1}{r}-\frac{1}{q})} \mathcal{L}^{2(d-1)}(F_{j,k,l})^{\frac{1}{q}}.$$

Let $\epsilon_1, \epsilon_2 > 0$ and $S_{k,l} = 2^{k\epsilon_1 - l(\frac{1}{r}-\frac{1}{q})\epsilon_2}$. Then

$$2^{k+l(\frac{1}{r}-\frac{1}{q})} \mathcal{L}^{2(d-1)}(F_{j,k,l})^{\frac{1}{q}} = S_{k,l}\left(2^k 2^{l(\frac{1}{r}-\frac{1}{q})\frac{1+\epsilon_2}{1-\epsilon_1}} \mathcal{L}^{2(d-1)}(F_{j,k,l})^{\frac{1}{q(1-\epsilon_1)}}\right)^{1-\epsilon_1}.$$

Provided that ϵ_2 is sufficiently small relative to ϵ_1, we have

$$\begin{aligned} \left(\frac{1}{r} - \frac{1}{q}\right)\frac{1+\epsilon_2}{1-\epsilon_1} &= \frac{1}{r} - \frac{1}{q(1-\epsilon_1)} + \frac{\epsilon_1}{r(1-\epsilon_1)} + \epsilon_2\left(\frac{1}{r(1-\epsilon_1)} - \frac{1}{q(1-\epsilon_1)}\right) \\ &\geq \frac{1}{r} - \frac{1}{q(1-\epsilon_1)}. \end{aligned} \tag{3.13}$$

We then have

$$2^k 2^{l(\frac{1}{r}-\frac{1}{q})\frac{1+\epsilon_2}{1-\epsilon_1}} \mathcal{L}^{2(d-1)}(F_{j,k,l})^{\frac{1}{q(1-\epsilon_1)}} \lesssim 2^k 2^{l(\frac{1}{r}-\frac{1}{q(1-\epsilon_1)})} \mathcal{L}^{2(d-1)}(F_{j,k,l})^{\frac{1}{q(1-\epsilon_1)}}$$
$$\lesssim \mathcal{L}^d(E_j)^{\frac{1}{p}},$$

where the first inequality follows from (3.13) and the second inequality follows from (3.11) and the fact that $\frac{\mathcal{L}^{2(d-1)}(F_{j,k,l})}{2^l} \lesssim 1$. Since $S_{k,l}$ is summable this gives

$$\begin{aligned}
\|T[f]\|_{L^q(L^r),K} &\lesssim \sum_{j=-\infty}^{\infty} 2^j \mathcal{L}^d(E_j)^{\frac{1-\epsilon_1}{p}} \\
&= \sum_{j=-\infty}^{0} 2^{j\epsilon_1} \left(2^j \mathcal{L}^d(E_j)^{\frac{1}{p}}\right)^{1-\epsilon_1} + \sum_{j=1}^{\infty} 2^{-j\epsilon_1} \left(2^j \mathcal{L}^d(E_j)^{\frac{1-\epsilon_1}{p(1+\epsilon_1)}}\right)^{1+\epsilon_1} \\
&\lesssim \|f\|_{L^p}^{1-\epsilon_1} + \|f\|_{L^{\frac{p(1+\epsilon_1)}{1-\epsilon_1}}}^{1+\epsilon_1} \\
&\lesssim \|f\|_{L^{\frac{p(1+\epsilon_1)}{1-\epsilon_1}}}^{1-\epsilon_1} + \|f\|_{L^{\frac{p(1+\epsilon_1)}{1-\epsilon_1}}}^{1+\epsilon_1} \\
&= 2\|f\|_{L^{p+\epsilon}},
\end{aligned}$$

where we choose ϵ_1 solving $p\frac{1+\epsilon_1}{1-\epsilon_1} = p + \epsilon$, and recall (3.12) for the last equation. □

3.2 The two-ends reduction

In order to obtain a favorable value of p in Theorem 3.1, we will employ a version of the *two-ends reduction* from [43], namely that it suffices to assume for $\lambda \leq \rho \leq \lambda^\epsilon$, $(\xi, x) \in F$, $z \in \mathbb{R}^d$, that

$$\int_0^1 \chi_{E\cap B(z,\rho)}(x + t(\xi + e_d))\, dt \lesssim \lambda \rho^{B\epsilon} \tag{3.14}$$

where $B = \frac{1}{4p}$. To justify this reduction, we will use induction on scales via Claim 3.5 below.

Claim 3.5. *Suppose that* $1 + \frac{d-1}{r} \geq \frac{d}{p}$ *and that the inequality*

$$\mathcal{L}^d(E) \geq C_\epsilon \lambda^{p+\epsilon} \mathcal{L}^{2(d-1)}(F)^{\frac{p}{q}} \Omega(F)^{p\left(\frac{1}{r}-\frac{1}{q}\right)} \tag{3.15}$$

holds for

$$E \subset Q, \;\; \lambda \geq \lambda_0, \;\; \text{and } E, F, \lambda \text{ satisfying (3.10).} \tag{3.16}$$

Then, we also have for $\rho < 1$

$$\mathcal{L}^d(\tilde{E}) \geq \rho^{-\epsilon} C_\epsilon \tilde{\lambda}^{p+\epsilon} \mathcal{L}^{2(d-1)}(\tilde{F})^{\frac{p}{q}} \Omega(\tilde{F})^{p\left(\frac{1}{r}-\frac{1}{q}\right)} \tag{3.17}$$

when $\tilde{E}$ *is contained in any sub-cube* $\tilde{Q} \subset Q$ *with side length* ρ*;* $\tilde{\lambda} \geq \rho\lambda_0$*; and* $\tilde{E}, \tilde{F}, \tilde{\lambda}$ *are as in (3.10) (i.e.* $T[\chi_{\tilde{E}}] \geq \tilde{\lambda}$ *on* $\tilde{F}$*).*

Proof. The proof is simply a change of variables, mapping $\tilde{Q}$ to Q. Let z be the center of $\tilde{Q}$. Then, letting $t_z e_d = \text{proj}_{e_d}(z)$, $x_z = \text{proj}_H(z)$, and $\tilde{t} = \frac{1}{\rho}\left(t - \left(t_z - \frac{\rho}{2}\right)\right)$, we see that

$$\begin{aligned}
T[\chi_{\tilde{E}}](\xi, x) &= \int_{t_z-\frac{\rho}{2}}^{t_z+\frac{\rho}{2}} \chi_{\tilde{E}}(x + t(\xi + e_d))\, dt \\
&= \int_{t_z-\frac{\rho}{2}}^{t_z+\frac{\rho}{2}} \chi_{\tilde{E}-z+\frac{\rho}{2}e_d}\left(x - x_z + t(\xi+e_d) - \left(t_z - \frac{\rho}{2}\right)e_d\right)\, dt \\
&= \rho \int_0^1 \chi_{\tilde{E}-z+\frac{\rho}{2}e_d}\left(x - x_z + \left(t_z - \frac{\rho}{2}\right)\xi + \rho\tilde{t}\,(\xi + e_d)\right)\, d\tilde{t} \\
&= \rho \int_0^1 \chi_{\frac{1}{\rho}\left(\tilde{E}-z+\frac{\rho}{2}e_d\right)}\left(\frac{1}{\rho}\left(x - x_z + \left(t_z - \frac{\rho}{2}\right)\xi\right) + \tilde{t}(\xi+e_d)\right)\, d\tilde{t} \\
&= \rho T\left[\chi_E\right]\left(\xi, \frac{1}{\rho}\left(x - x_z + \left(t_z - \frac{\rho}{2}\right)\xi\right)\right),
\end{aligned}$$

where $E = \frac{1}{\rho}\left(\tilde{E} - z + \frac{\rho}{2}e_d\right)$.

Let $F = \{(\xi, \frac{1}{\rho}\left(x - x_z + \left(t_z - \frac{\rho}{2}\right)\xi\right)) : (\xi, x) \in \tilde{F}\}$ and $\lambda = \frac{1}{\rho}\tilde{\lambda}$, so that $T[\chi_E] \geq \lambda$

on F. Then

$$\mathcal{L}^d(E) = \rho^{-d}\mathcal{L}^d(\tilde{E}),$$

$$\mathcal{L}^{2(d-1)}(F) = \rho^{-(d-1)}\mathcal{L}^{2(d-1)}\tilde{F},$$

and

$$\Omega(F) = \rho^{-(d-1)}\Omega(\tilde{F}).$$

By construction, E is contained in Q and, since $\tilde{\lambda} \geq \rho\lambda_0$, we have $\lambda \geq \lambda_0$. Thus, we may apply (3.15), obtaining

$$\begin{aligned} \rho^{-d}\mathcal{L}^d(\tilde{E}) &\geq C_\epsilon \left(\rho^{-1}\tilde{\lambda}\right)^{p+\epsilon} \left(\rho^{-(d-1)}\mathcal{L}^{2(d-1)}(\tilde{F})\right)^{\frac{p}{q}} (\rho^{-(d-1)}\Omega(\tilde{F}))^{p\left(\frac{1}{r}-\frac{1}{q}\right)} \\ &= \rho^{-p(1+\frac{d-1}{r})}\rho^{-\epsilon}C_\epsilon\tilde{\lambda}^{p+\epsilon}\mathcal{L}^{2(d-1)}(\tilde{F})^{\frac{p}{q}}\Omega(\tilde{F})^{p\left(\frac{1}{r}-\frac{1}{q}\right)}. \end{aligned}$$

Then, since $1 + \frac{d-1}{r} \geq \frac{d}{p}$ and $\rho < 1$, we have $\rho^{d-p(1+\frac{d-1}{r})} > 1$ and thus our conclusion (3.17) holds. □

We will use induction on λ_0 to show that proving (3.15) for arbitrary λ under the two-ends reduction is sufficient to prove (3.15) for arbitrary λ in general.

Note that (3.14) holds trivially for any fixed choice of λ by a sufficiently large choice of the implicit constant. Thus, we may start our induction and assume that (3.15) holds for an initial λ_0. Now, suppose that (3.15) is known to hold with $\lambda_0 = \Lambda \leq 64^{-\frac{1}{\epsilon}}$ and that we want to prove it with $\lambda_0 = \frac{1}{2}\Lambda$. Let $\frac{1}{2}\Lambda \leq \lambda \leq \Lambda$, and let E, F be as in (3.16). Let $\overline{F}$ be the subset of F for which the two ends condition (3.14) fails. If

$$\mathcal{L}^{2(d-1)}(\overline{F}) \leq \frac{1}{2}\mathcal{L}^{2(d-1)}(F), \tag{3.18}$$

then we may apply our knowledge of (3.15) under the two-ends reduction to the set $F \setminus \overline{F}$. Thus, without loss of generality we assume the inequality opposite to (3.18).

By dyadic pigeonholing we may find $\rho_0 \in [\lambda, \lambda^\epsilon]$ and $\hat{F}$ so that (3.14) fails with $\rho = \rho_0$ for $(\xi, x) \in \hat{F}$ and

$$\mathcal{L}^{2(d-1)}(\hat{F}) \gtrsim \frac{\mathcal{L}^{2(d-1)}(F)}{|\log(\lambda)|}.$$

We then tile Q by a collection $\{Q_j\}$ of cubes with side length ρ_0, and let $\tilde{E}_j = E \cap \tilde{Q}_j$ where $\tilde{Q}_j$ is the cube with the same center as Q_j and side length $4\rho_0$. For each $(\xi, x) \in \hat{F}$, (3.14) fails for some $z_{\xi,x} \in Q_{j_{\xi,x}}$, giving

$$T[\chi_{\tilde{E}_{j_{\xi,x}}}](\xi, x) \geq \lambda \rho_0^{B\epsilon}. \tag{3.19}$$

Henceforth, we take $\tilde{\lambda} = \lambda \rho_0^{B\epsilon}$ and let $\tilde{F}_j = \{(\xi, x) \in \hat{F} : j = j_{\xi,x}\}$. Without loss of generality, we assume that the constant on the right hand side of (3.14) is at least 1 and thus we may ignore it in (3.19).

For each j we now have $T[\chi_{\tilde{E}_j}] \geq \tilde{\lambda}$ on $\tilde{F}_j$, and $\tilde{E}_j$ contained in a sub-cube of Q with side length $4\rho_0$. Naturally, we will want to apply Claim 3.5 in order to estimate the size of the $\tilde{E}_j$. In order to apply the claim, it only remains to verify that $\tilde{\lambda} \geq 4\rho_0\Lambda$. In fact, assuming without loss of generality that $\epsilon \leq \frac{1}{2B}$, we have

$$\tilde{\lambda} \geq \lambda \rho_0^{B\epsilon} = \rho_0 \Lambda \frac{\lambda}{\Lambda} \rho_0^{B\epsilon - 1} \geq \rho_0 \Lambda \frac{1}{2} \rho_0^{B\epsilon - 1} \geq \rho_0 \Lambda \frac{1}{2} \rho_0^{-\frac{1}{2}} \geq \rho_0 \Lambda \frac{1}{2} \Lambda^{-\frac{1}{2}\epsilon} \geq 4\rho_0 \Lambda.$$

Thus, we may apply Claim 3.5. We then have by (3.17)

$$\begin{aligned} \mathcal{L}^d(\tilde{E}_j) &\geq (4\rho_0)^{-\epsilon} C_\epsilon \tilde{\lambda}^{p+\epsilon} \mathcal{L}^{2(d-1)}(\tilde{F}_j)^{\frac{p}{q}} \Omega(\tilde{F}_j)^{p\left(\frac{1}{r} - \frac{1}{q}\right)} \\ &\geq \frac{1}{4} \rho_0^{-\epsilon + (p+\epsilon)B\epsilon} C_\epsilon \lambda^{p+\epsilon} \mathcal{L}^{2(d-1)}(\tilde{F}_j)^{\frac{p}{q}} \Omega(\tilde{F}_j)^{p\left(\frac{1}{r} - \frac{1}{q}\right)}. \end{aligned}$$

Recalling that $q \leq r$, we observe that, since $\tilde{F}_j \subset F$, we have $\Omega(\tilde{F}_j)^{p(\frac{1}{r} - \frac{1}{q})} \geq$

$\Omega(F)^{p(\frac{1}{r}-\frac{1}{q})}$. Since the $\tilde{E}_j$ are finitely overlapping and $q \geq p$, we then have

$$\begin{aligned}\mathcal{L}^d(E) &\gtrsim \sum_j \mathcal{L}^d(\tilde{E}_j) \qquad (3.20)\\ &\geq \frac{1}{4}\rho_0^{-\epsilon+(p+\epsilon)B\epsilon} C_\epsilon \lambda^{p+\epsilon} \sum_j \mathcal{L}^{2(d-1)}(\tilde{F}_j)^{\frac{p}{q}} \Omega(\tilde{F}_j)^{p\left(\frac{1}{r}-\frac{1}{q}\right)}\\ &\geq \frac{1}{4}\rho_0^{-\epsilon+(p+\epsilon)B\epsilon} C_\epsilon \lambda^{p+\epsilon} \Omega(F)^{p\left(\frac{1}{r}-\frac{1}{q}\right)} \sum_j \mathcal{L}^{2(d-1)}(\tilde{F}_j)^{\frac{p}{q}}\\ &\geq \frac{1}{4}\rho_0^{-\epsilon+(p+\epsilon)B\epsilon} C_\epsilon \lambda^{p+\epsilon} \Omega(F)^{p\left(\frac{1}{r}-\frac{1}{q}\right)} \mathcal{L}^{2(d-1)}(\hat{F})^{\frac{p}{q}}\\ &\gtrsim \rho_0^{-\epsilon+(p+\epsilon)B\epsilon} |\log(\lambda)|^{-\frac{p}{q}} C_\epsilon \lambda^{p+\epsilon} \Omega(F)^{p\left(\frac{1}{r}-\frac{1}{q}\right)} \mathcal{L}^{2(d-1)}(F)^{\frac{p}{q}}.\end{aligned}$$

Recalling that $B = \frac{1}{4p}$ and assuming that $\epsilon < p$, we have

$$\rho_0^{-\epsilon+(p+\epsilon)B\epsilon} |\log(\lambda)|^{-\frac{p}{q}} \geq \lambda^{-\frac{1}{2}\epsilon^2} |\log(\lambda)|^{-\frac{p}{q}}. \qquad (3.21)$$

It only remains to verify that the right side of (3.21) is large enough to overcome the implicit constant in (3.20). Noting that this constant is independent of the constant in (3.14), we see that this may be accomplished by a sufficiently small initial choice of λ_0.

3.3 The main estimate

For $t \in \mathbb{R}$, let H_t denote the plane $H + te_d$. Given a line g which intersects H in exactly one point, we define

$$\pi_t(g) = H_t \cap g.$$

For collections G of such lines, we are interested in lower bounds for the size of $\pi_t(G)$ in terms of the size of G.

We first consider the purely combinatorial setting where G is finite. Then, from the fact that a line is determined by two points, we see that for every $t_1 \neq t_2$,

$$\#G^{\frac{1}{2}} \leq \sup_{t=t_1,t_2} \#\pi_t(G)$$

where # denotes cardinality. In [20], Katz and Tao showed that if $t_0, t_\infty, t_1, t_{1'}, t_2, t_{2'}$ satisfy a certain algebraic condition, and if the lines in G point in distinct directions, then

$$\#G^{\frac{4}{7}} \lesssim \sup_{t=t_0,t_\infty,t_1,t_{1'},t_2,t_{2'}} \#\pi_t(G). \tag{3.22}$$

In order to give a bound of type (3.3), one must consider the more general case where the lines in G do not point in distinct directions. Suppose that at most M lines in G point in each direction. Then by taking a maximal direction separated subset of G, the estimate

$$\#G^{\frac{4}{7}} \lesssim M^{\frac{4}{7}} \sup_{t=t_0,t_\infty,t_1,t_{1'},t_2,t_{2'}} \#\pi_t(G) \tag{3.23}$$

follows trivially from (3.22). However, following the proof of (3.22) with the quantity M in mind, one actually obtains

$$\#G^{\frac{4}{7}} \lesssim M^{\frac{1}{7}} \sup_{t=t_0,t_\infty,t_1,t_{1'},t_2,t_{2'}} \#\pi_t(G) \tag{3.24}$$

with no additional arguments required. This is, in fact, the sharp power of M for the given power of G, as one may verify by letting G_n be the set of lines determined by the pairs of points $(ie_1, ie_1 + je_1 + e_d)$ where $1 \leq i, j \leq n$, letting $t_0, t_\infty, t_1, t_{1'}, t_2, t_{2'}$ be any fixed rational numbers, and considering n large.

To prove the desired operator estimates, one must make the transition from the combinatorial to the continuous setting. The maximal operator bound in [20] was proven using a δ-discretization argument and a refinement of (3.22) which took into account possible δ-uncertainties. It is possible to adapt (3.24) for use with a discretization argument and use this estimate to prove Theorem 3.1. Instead, we will prove the analog of (3.24) for Lebesgue-measurable sets of lines and avoid discretization entirely.

Let G be a set of lines in $\mathbb{R}^d$, each of which intersect the plane H in exactly one point. Considering the coordinates for the X-ray transform T, we parametrize G by the subset

$$G_X = \{(\xi, x) : \text{ there exists } g \in G \text{ with } x, (x + \xi + e_d) \in g\}$$

of $H \times H$. For $t_1 \neq t_2$, we may also parametrize G by the subset

$$G_{t_1,t_2} = \{(\pi_{t_1}(g), \pi_{t_2}(g)) : g \in G\}$$

of $H \times H$. Using the "line property"

$$(x, y) \in G_{t_1,t_2} \Leftrightarrow \left(\frac{t_2 - t_3}{t_2 - t_1}x + \frac{t_3 - t_1}{t_2 - t_1}y, \frac{t_2 - t_4}{t_2 - t_1}x + \frac{t_4 - t_1}{t_2 - t_1}y\right) \in G_{t_3,t_4}, \tag{3.25}$$

we may change variables to give

$$|G| := \mathcal{L}^{2(d-1)}(G_X) = \mathcal{L}^{2(d-1)}(G_{0,1}) = |t_1 - t_2|^{-(d-1)}\mathcal{L}^{2(d-1)}(G_{t_1,t_2}). \tag{3.26}$$

Henceforth, we will use the abbreviation

$$D_{t,t'} := |t - t'|^{-(d-1)}.$$

Finally, define

$$\Omega(G) := \Omega(G_X) = \sup_{\xi \in H} \mathcal{L}^{d-1}(\{x : (\xi, x) \in G_X\}).$$

In practice, each line in G will intersect the cube Q, and $G_X \subset B(0, C) \times \mathbb{R}^{d-1}$. Thus $\Omega(G), |G|, \mathcal{L}^{d-1}(\pi_t(G)) < \infty$. We will always assume that this inequality holds below. To put the following proposition in context, we point out that, as we will see later, the quantity $\frac{k}{\alpha-\beta}$ in an estimate of the form (3.27) corresponds to the quantity $\frac{r}{p}$ in an estimate of the form (3.3).

Proposition 3.6. *Suppose that for $t_1, \dots, t_k \in \mathbb{R}$, we have*

$$|G|^\alpha \lesssim C_{t_1,\dots,t_k} \Omega(G)^\beta \prod_{i=1,\dots,k} \mathcal{L}^{d-1}(\pi_{t_i}(G)) \tag{3.27}$$

for each set of lines G, where $0 \leq \beta$ and $\alpha \leq k$.

Then, for any $t_0, t_\infty, t_{1'}, \dots, t_{k'} \in \mathbb{R}$ satisfying $t_0 \neq t_\infty$;

$$t_i \neq t_0, \quad t_{i'} \neq t_0, \quad t_i \neq t_\infty, \text{ for } i = 1, \dots, k;$$

and satisfying the requirement that

$$s = s_{t_0,t_\infty}(t_i, t_{i'}) := (t_\infty - t_0) \frac{t_i - t_0}{(t_i - t_\infty)(t_{i'} - t_0)} \tag{3.28}$$

is independent of i for $i = 1, \dots, k$, we have

$$\begin{aligned} |G|^{2k} \lesssim & \left(C_{t_1,\dots,t_k} D_{t_0,t_\infty}^{k-\alpha} \prod_{i=1,\dots k} D_{t_{i'},t_0} \right) \Omega(G)^{\beta+k-\alpha} \\ & \cdot \mathcal{L}^{d-1}(\pi_{t_\infty}(G))^{k-\alpha} \mathcal{L}^{d-1}(\pi_{t_0}(G))^k \prod_{i=1,\dots k} \mathcal{L}^{d-1}(\pi_{t_i}(G)) \mathcal{L}^{d-1}(\pi_{t_{i'}}(G)). \end{aligned} \tag{3.29}$$

For $t_1 \neq t_2$, the trivial estimate

$$|G| \leq D_{t_1,t_2} \mathcal{L}^{d-1}(\pi_{t_1}(G)) \mathcal{L}^{d-1}(\pi_{t_2}(G)) \tag{3.30}$$

follows directly from (3.26). Thus, applying Proposition 3.6 with $k = 2, \alpha = 1, \beta = 0$, and $C_{t_1,t_2} = D_{t_1,t_2}$, we obtain

Corollary 3.7. *Let $t_1, t_{1'}, t_2, t_{2'}, t_0, t_\infty \in \mathbb{R}$ satisfy*

$$t_1 \neq t_2; \; t_0 \neq t_\infty, t_1, t_{1'}, t_2, t_2'; \; t_\infty \neq t_1, t_2;$$

and the requirement that s is independent of i in (3.28). Then

$$|G|^4 \lesssim \left(D_{t_1,t_2} D_{t_0,t_\infty} D_{t_{1'},t_0}, D_{t_2',t_0} \right) \Omega(G) \sup_{t = t_\infty, t_0, t_1, t_{1'}, t_2, t_{2'}} \mathcal{L}^{d-1}(\pi_t(G))^7. \tag{3.31}$$

In Section 3.4, we will use a uniformization argument to obtain Theorem 3.1 from Corollary 3.7. In Section 3.5, we will consider further iterations of Proposition 3.6.

Proof of Proposition 3.6. We will begin by defining the set

$$V = \{(g_1, g_2) \in G \times G : \pi_{t_0}(g_1) = \pi_{t_0}(g_2)\}.$$

For any $w = (g_1, g_2) \in V$, let $\gamma_i(w) = g_i$ when $i = 1, 2$. Consider the function on V,

$$\nu = s\pi_{t_\infty}(\gamma_1) + \pi_{t_\infty}(\gamma_2) - \pi_{t_0}(\gamma_2).$$

Our purpose in defining ν is to obtain the equivalence classes in subsets W of V determined by the fibers of ν:

$$W_{\nu_0} := W \cap \nu^{-1}(\nu_0) \text{ for } \nu_0 \in H.$$

Let $G_{\nu_0} = \gamma_1(V_{\nu_0})$. Then we may calculate an upper bound for $|G_{\nu_0}|$ in terms of $\Omega(G)$ and $\mathcal{L}^{d-1}(\pi_{t_\infty}(G))$. First note that

$$\begin{aligned} |G_{\nu_0}| &= D_{t_0,t_\infty} \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}^{d-1}} \chi_{(G_{\nu_0})_{t_0,t_\infty}}(x, y)\ dx\ dy \\ &= D_{t_0,t_\infty} \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}^{d-1}} \chi_{G_{t_0,t_\infty}}(x, y)\chi_{G_{t_0,t_\infty}}(x, \nu_0 - sy + x)\ dx\ dy. \end{aligned} \tag{3.32}$$

Fix y and let $\xi_y = \frac{\nu_0 - sy}{t_\infty - t_0}$. Observe that

$$\begin{aligned} \int_{\mathbb{R}^{d-1}} \chi_{G_{t_0,t_\infty}}(x, x + \nu_0 - sy)\ dx &= \int_{\mathbb{R}^{d-1}} \chi_{G_{t_0,t_\infty}}(x, x + (t_\infty - t_0)\xi_y)\ dx \\ &= \int_{\mathbb{R}^{d-1}} \chi_{G_{0,1}}(x - t_0\xi_y, x - t_0\xi_y + \xi_y)\ dx \\ &= \mathcal{L}^{d-1}(\{x : (\xi_y, x) \in G_X\}) \\ &\le \Omega(G), \end{aligned} \tag{3.33}$$

where the second equation follows from the line property (3.25), and the third equation follows from the translation invariance of $\mathcal{L}^{d-1}$. Combining (3.32) and (3.33), we have

$$|G_{\nu_0}| \leq D_{t_0,t_\infty}\mathcal{L}^{d-1}(\pi_{t_\infty}(G))\Omega(G). \tag{3.34}$$

The remainder of our argument will consist of using (3.27) to obtain a lower bound for $|G_{\nu_0}|$ when ν_0 is chosen favorably. For any $t, t' \neq t_0$ and subset W of V, we may parametrize W by the subset

$$W_{t_0,t,t'} = \{(\pi_{t_0}(g_1), \pi_t(g_1), \pi_{t'}(g_2)) : (g_1, g_2) \in W\}$$

of H^3. Using our line-property (3.25), we note that, as was the case with $|G|$,

$$|W| := D_{t_0,t}D_{t_0,t'}\mathcal{L}^{3(d-1)}(W_{t_0,t,t'})$$

is independent of t, t'.

For any $t, t' \neq t_0$, $W \subset V$, we may consider the subset of W which is "popular" with respect to the double projection $(\pi_t(\gamma_1), \pi_{t'}(\gamma_2))$

$$W_{t_0,t,t'}^{\langle \pi_{t\otimes t'}\rangle} := \left\{(x,y,z) : D_{t_0,t}D_{t_0,t'}\int_{\mathbb{R}^{d-1}} \chi_{W_{t_0,t,t'}}(x',y,z)\ dx' \geq \frac{|W|}{2\mathcal{L}^{d-1}(\pi_t(G))\mathcal{L}^{d-1}(\pi_{t'}(G))}\right\}. \tag{3.35}$$

After estimating $\left|W \setminus W^{\langle \pi_{t\otimes t'}\rangle}\right|$, one observes that

$$|W^{\langle \pi_{t\otimes t'}\rangle}| \geq \frac{1}{2}|W|. \tag{3.36}$$

Thus, abbreviating

$$V' := \left(\left(\left(V^{\langle \pi_{t_1\otimes t_1'}\rangle}\right)^{\langle \pi_{t_2\otimes t_2'}\rangle}\right)\ldots\right)^{\langle \pi_{t_k\otimes t_k'}\rangle},$$

we have

$$|V'| \gtrsim |V|. \tag{3.37}$$

Given any subset W of V,

$$
\begin{aligned}
|W| &= D_{t_0,t_\infty} D_{t_0,t_\infty} \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}^{d-1}} \chi_{W_{t_0,t_\infty,t_\infty}}(x,y,z)\ dx\ dy\ dz \\
&= D_{t_0,t_\infty} D_{t_0,t_\infty} \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}^{d-1}} \chi_{W_{t_0,t_\infty,t_\infty}}(x,y,\nu' - sy + x)\ dx\ dy\ d\nu' \\
&= D_{t_0,t_\infty} D_{t_0,t_\infty} \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}^{d-1}} \chi_{(\gamma_1(W_{\nu'}))_{t_0,t_\infty}}(x,y)\ dx\ dy\ d\nu'.
\end{aligned} \tag{3.38}
$$

Substituting (3.38) with $W = V'$ and $W = V$ into the left and right hand sides respectively of (3.37), we observe that

$$
|G'_{\nu_0}| \gtrsim |G_{\nu_0}| \tag{3.39}
$$

for some $\nu_0 \in \nu(V)$, where we define $G'_{\nu_0} = \gamma_1\left((V')_{\nu_0}\right)$.

Applying our hypothesis (3.27) to the set of lines G'_{ν_0}, we obtain

$$
\begin{aligned}
|G'_{\nu_0}|^\alpha &\lesssim C_{t_1,\ldots,t_k} \Omega(G'_{\nu_0})^\beta \prod_{i=1,\ldots,k} \mathcal{L}^{d-1}(\pi_{t_i}(G'_{\nu_0})) \\
&\le C_{t_1,\ldots,t_k} \Omega(G)^\beta \prod_{i=1,\ldots,k} \mathcal{L}^{d-1}(\pi_{t_i}(G'_{\nu_0})),
\end{aligned} \tag{3.40}
$$

where the second inequality follows from the fact that $G'_{\nu_0} \subset G$ and the condition that $\beta \ge 0$.

Suppose $y \in \pi_{t_i}(G'_{\nu_0})$. Then $y = \pi_{t_i}(g_1)$ where $(g_1, g_2) \in (V')_{\nu_0}$. Letting $z = \pi_{t_{i'}}(g_2)$, we have by definition of V'

$$
\frac{|V|}{\mathcal{L}^{d-1}(\pi_{t_i}(G))\mathcal{L}^{d-1}(\pi_{t_{i'}}(G))} \lesssim D_{t_i,t_0} D_{t_{i'},t_0} \int_{\mathbb{R}^{d-1}} \chi_{V_{t_0,t_i,t_{i'}}}(x,y,z)\ dx. \tag{3.41}
$$

We will now need to use our definition of s, rewriting

$$\begin{aligned}\nu &= s\left(\pi_{t_\infty}(\gamma_1) - \pi_{t_0}(\gamma_1)\right) + \left(\pi_{t_\infty}(\gamma_2) - \pi_{t_0}(\gamma_2)\right) + s\pi_{t_0}(\gamma_2) \\ &= s\frac{t_\infty - t_0}{t_i - t_0}\left(\pi_{t_i}(\gamma_1) - \pi_{t_0}(\gamma_1)\right) + \frac{t_\infty - t_0}{t_{i'} - t_0}\left(\pi_{t_{i'}}(\gamma_2) - \pi_{t_0}(\gamma_2)\right) + s\pi_{t_0}(\gamma_2) \\ &= s\frac{t_\infty - t_0}{t_i - t_0}\pi_{t_i}(\gamma_1) + \frac{t_\infty - t_0}{t_{i'} - t_0}\pi_{t_{i'}}(\gamma_2) + \pi_{t_0}(\gamma_2)\left(\frac{t_0 - t_\infty}{t_{i'} - t_0} + s\frac{t_i - t_\infty}{t_i - t_0}\right) \\ &= s\frac{t_\infty - t_0}{t_i - t_0}\pi_{t_i}(\gamma_1) + \frac{t_\infty - t_0}{t_{i'} - t_0}\pi_{t_{i'}}(\gamma_2),\end{aligned} \tag{3.42}$$

where $\pi_{t_0}(\gamma_j)$ is independent of j by definition of V, and where the last equation follows from (3.28). The point is that membership in V_{ν_0} is determined by the double projection $(\pi_{t_i}(\gamma_1), \pi_{t_i'}(\gamma_2))$. Hence

$$\begin{aligned}\int_{\mathbb{R}^{d-1}} \chi_{V_{t_0,t_i,t_{i'}}}(x,y,z)\ dx &= \int_{\mathbb{R}^{d-1}} \chi_{(V_{\nu_0})_{t_0,t_i,t_{i'}}}(x,y,z)\ dx \\ &= \int_{\mathbb{R}^{d-1}} \chi_{(G_{\nu_0})_{t_0,t_i}}(x,y)\ dx.\end{aligned} \tag{3.43}$$

Combining (3.41) and (3.43), we obtain

$$\begin{aligned}\mathcal{L}^{d-1}(\pi_{t_i}(G'_{\nu_0})) &= \int_{\mathbb{R}^{d-1}} \chi_{\pi_{t_i}(G'_{\nu_0})}(y)\ dy \\ &\lesssim D_{t_i,t_0} D_{t_{i'},t_0} \frac{\mathcal{L}^{d-1}(\pi_{t_i}(G))\mathcal{L}^{d-1}(\pi_{t_{i'}}(G))}{|V|} \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}^{d-1}} \chi_{(G_{\nu_0})_{t_0,t_i}}(x,y)\ dx\ dy \\ &= |G_{\nu_0}| D_{t_{i'},t_0} \frac{\mathcal{L}^{d-1}(\pi_{t_i}(G))\mathcal{L}^{d-1}(\pi_{t_{i'}}(G))}{|V|}.\end{aligned} \tag{3.44}$$

Combining (3.39), (3.40), and (3.44), we have

$$|V|^k \lesssim C_{t_1,\ldots,t_k,\tau} \Omega(G)^\beta |G_{\nu_0}|^{k-\alpha} \prod_{i=1,\ldots,k} D_{t_0,t_i'} \mathcal{L}^{d-1}(\pi_{t_i}(G))\mathcal{L}^{d-1}(\pi_{t_{i'}}(G)). \tag{3.45}$$

From the Cauchy-Schwarz inequality, we obtain a lower bound for $|V|$ in terms of

$|G|$:

$$\left(\frac{|G|}{D_{t_0,t}}\right)^2 \leq \mathcal{L}^{d-1}(\pi_{t_0}(G)) \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}^{d-1}} \chi_{G_{t_0,t}}(x,y)\ dy \int_{\mathbb{R}^{d-1}} \chi_{G_{t_0,t}}(x,z)\ dz\ dx$$
$$= \mathcal{L}^{d-1}(\pi_{t_0}(G)) \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}^{d-1}} \chi_{V_{t_0,t,t}}(x,y,z)\ dx\ dy\ dz$$
$$= \mathcal{L}^{d-1}(\pi_{t_0}(G)) \frac{|V|}{D_{t_0,t}^2},$$

which simplifies to

$$|V| \geq \frac{|G|^2}{\mathcal{L}^{d-1}(\pi_{t_0}(G))}. \tag{3.46}$$

Combining (3.46), (3.45), and (3.34), we finally obtain (3.29). □

3.4 Uniformization

We now want to find six suitably "average" slices of E to which we will apply Corollary 3.7, allowing us to prove (3.15) under the two-ends reduction. Our argument below is a continuous version of the uniformization argument in [20]. Let E, F, λ be as in (3.16), and define

$$\begin{aligned}\mathfrak{M}(E,F) &= \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}^{d-1}} \chi_F(\xi,x) T[\chi_E](\xi,x)\ dx\ d\xi \\ &= \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}^{d-1}} \int_0^1 \chi_F(\xi,x)\chi_E(x+t(\xi+e_d))\ dt\ dx\ d\xi \\ &= \int_{\mathbb{R}^d} \chi_E(z) \int_{\mathbb{R}^{d-1}} \chi_F(\xi, x_z - t_z\xi)\ d\xi\ dz,\end{aligned}$$

where $x_z = \mathrm{proj}_H(z)$ and $t_z e_d = \mathrm{proj}_{e_d}(z)$. By definition of F, $\mathfrak{M}(E,F) \geq \lambda \mathcal{L}^{2(d-1)}(F)$.

We will abbreviate $\gamma_d = \mathrm{proj}_{e_d}$. Let

$$S_0 = \{t \in [0,1] : \mathcal{L}^{d-1}(E \cap \gamma_d^{-1}(t)) \ll \lambda \mathcal{L}^{2(d-1)}(F)\}$$

and for $k > 0$ let

$$S_k = \{t \in [0,1] : (\lambda \mathcal{L}^{2(d-1)}(F))^{1-(k-1)\epsilon} \lesssim \mathcal{L}^{d-1}(E \cap \gamma_d^{-1}(t)) \ll (\lambda \mathcal{L}^{2(d-1)}(F))^{1-k\epsilon}\}.$$

Recalling that $F \subset B(0, C) \times \mathbb{R}^{d-1}$, we note that

$$\int_{\mathbb{R}^{d-1}} \chi_F(\xi, \mathrm{proj}_H(z) - \mathrm{proj}_{e_d}(z)\xi)\ d\xi \lesssim 1$$

for every z. Hence, defining $E_k = E \cap \gamma_d^{-1}(S_k)$, we have $\mathfrak{M}(E_0, F) \ll \lambda \mathcal{L}^{2(d-1)}(F)$. Thus, since $E = \bigcup_{k \lesssim \frac{1}{\epsilon}} E_k$, an appropriate choice of implicit constants gives

$$\mathfrak{M}(E_{k_0}, F) \gtrsim \epsilon \lambda \mathcal{L}^{2(d-1)}(F) \gtrsim \lambda \mathcal{L}^{2(d-1)}(F)$$

for some $k_0 > 0$. Let $E' = E_{k_0}$, $S = S_{k_0}$, and

$$F' = \{(\xi, x) \in F : T[\chi_{E'}](\xi, x) \gtrsim \lambda\}.$$

Considering $\mathfrak{M}(E', F \setminus F')$, we note that

$$\mathfrak{M}(E', F') \gtrsim \lambda \mathcal{L}^{2(d-1)}(F).$$

We now proceed to find a point $(t_0, t_\infty, t_1, t_{1'}, t_2, t_{2'}) \in S^6$ with which we may apply Corollary 3.7 to our advantage. Due to the factors of D_{t_i,t_j} in (3.31), we would like to keep $|t_i - t_j|$ suitably large; this is facilitated by the two-ends reduction.

For every $(\xi, x) \in F'$, let

$$S_{\xi,x} = \{t' \in S : x + t'(\xi + e_d) \in E'\}.$$

Then, by definition of F',

$$\mu_{\xi,x} := \mathcal{L}^1(S_{\xi,x}) \gtrsim \lambda.$$

By (3.14) we have, for every $t \in \mathbb{R}$,

$$\mathcal{L}^1(\{t' \in S_{\xi,x} : |t'-t| < \lambda^\epsilon\}) \lesssim \lambda\lambda^{B\epsilon^2} \ll \lambda$$

where we assume, without loss of generality, that λ is sufficiently small to obtain the rightmost inequality. Thus,

$$\mathcal{L}^1(\{t' \in S_{\xi,x} : |t'-t| \geq \lambda^\epsilon\}) \gtrsim \mu_{\xi,x}$$

and so letting

$$P(\xi,x) = \{(t_0,t_\infty) \in (S_{\xi,x})^2 : |t_0 - t_\infty| \geq \lambda^\epsilon\},$$

we have $\mathcal{L}^2(P(\xi,x)) \gtrsim \mu_{\xi,x}^2$.

For each $(t_0,t_\infty) \in P(\xi,x)$ let

$$Q_{t_0,t_\infty}(\xi,x) = \{(t_1,t_{1'}) \in (S_{\xi,x})^2 : \text{ for all } \ i \neq j \in \{0,\infty,1,1'\}, \ |t_i - t_j| \geq \lambda^\epsilon\},$$

and note that

$$\mathcal{L}^2(Q_{t_0,t_\infty}(\xi,x)) \gtrsim \mu_{\xi,x}^2 \tag{3.47}$$

for every $(t_0,t_\infty) \in P(\xi,x)$.

We recall the definition of s from (3.28),

$$s_{t_0,t_\infty}(t_i,t_{i'}) := \frac{(t_\infty - t_0)(t_i - t_0)}{(t_i - t_\infty)(t_{i'} - t_0)}.$$

In order to satisfy the condition in Corollary 3.7 that s is independent of i, we consider the set

$$R_{t_0,t_\infty}(\xi,x) = \{(t_1,t_{1'},t_2,t_{2'}) \in Q_{t_0,t_\infty}(\xi,x)^2 : s_{t_0,t_\infty}(t_1,t_{1'}) = s_{t_0,t_\infty}(t_2,t_{2'})\}.$$

Below we abbreviate s_{t_0,t_∞} by s, $Q_{t_0,t_\infty}(\xi,x)$ by Q, $R_{t_0,t_\infty}(\xi,x)$ by R, and $\mu_{\xi,x}$ by μ. Also we use $\cdot \gtrapprox \cdot$ to denote $\cdot \gtrsim \lambda^{C\epsilon} \cdot$, and we similarly use $\lessapprox$ and $\approxeq$. We observe that $s(t_1,\cdot)$

is a diffeomorphism on an open set containing the support of $\chi_{Q(t_1,\cdot)}$, and so we may change variables to obtain

$$\begin{aligned}\mathcal{L}^2(Q) &= \int_{\mathbb{R}}\int_{\mathbb{R}} \chi_Q(t_1,t_{1'})\ dt_1\ dt_{1'} \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \chi_Q\left(t_1, u_{t_0,t_\infty}(t_1,s')\right)\left|\frac{(u_{t_0,t_\infty}(t_1,s')-t_0)^2(t_1-t_\infty)}{(t_1-t_0)(t_\infty-t_0)}\right|\ dt_1\ ds' \\ &\approxeq \int_{\mathbb{R}}\int_{\mathbb{R}} \chi_Q\left(t_1, u_{t_0,t_\infty}(t_1,s')\right)\ dt_1\ ds'\end{aligned} \tag{3.48}$$

where we define

$$u_{t_0,t_\infty}(t_1,s) = t_0 + \frac{(t_\infty-t_0)(t_1-t_0)}{s(t_1-t_\infty)}$$

and where the $\approxeq$ follows from the fact that the Jacobian is $\approxeq 1$ on Q. We apply Cauchy-Schwarz and change variables again to see that

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\mathbb{R}} \chi_Q\left(t_1, u_{t_0,t_\infty}(t_1,s')\right)\ dt_1\ ds' \\ &\le \mathcal{L}^1(s(Q))^{\frac{1}{2}}\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \chi_Q\left(t_1,u_{t_0,t_\infty}(t_1,s')\right)\chi_Q\left(t_2,u_{t_0,t_\infty}(t_2,s')\right)\ dt_1\ dt_2\ ds'\right)^{\frac{1}{2}} \\ &\lessapprox \left(\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \chi_Q\left(t_1,t_{1'}\right)\chi_Q\left(t_2,u_{t_0,t_\infty}(t_2,s(t_1,t_{1'}))\right)\ dt_1\ dt_{1'}\ dt_2\right)^{\frac{1}{2}}.\end{aligned} \tag{3.49}$$

Abbreviating

$$w_{t_0,t_\infty}(t_1,t_1',t_2) = u_{t_0,t_\infty}(t_2,s(t_1,t_{1'}))$$

we have by construction

$$s(t_1,t_{1'}) = s(t_2,w_{t_0,t_\infty}(t_1,t_{1'},t_2))$$

and hence

$$\chi_R(t_1,t_{1'},t_2,w_{t_0,t_\infty}(t_1,t_{1'},t_2)) = \chi_Q(t_1,t_{1'})\chi_Q(t_2,w_{t_0,t_\infty}(t_1,t_{1'},t_2)).$$

Thus, we combine (3.47), (3.48), and (3.49), to obtain

$$\begin{aligned} |R| &:= \int_{\mathbb{R}^3} \chi_R(t_1, t_{1'}, t_2, w_{t_0,t_\infty}(t_1, t_{1'}, t_2))\ dt_1\ dt_{1'}\ dt_2 \\ &\gtrapprox \mu^4. \end{aligned}$$

We have control over all of the $|t_i - t_j|$ relevant to Corollary 3.7, except for $|t_1 - t_2|$. However, since

$$\int_{\mathbb{R}^3} \chi_R(t_1, t_{1'}, t_2, w_{t_0,t_\infty}(t_1, t_{1'}, t_2))\chi_{[0,r]}(|t_1-t_2|)\ dt_1\ dt_{1'}\ dt_2 \lesssim r\mathcal{L}^1(\mathrm{proj}_{t_1}(R))\mathcal{L}^1(\mathrm{proj}_{t_{1'}}(R)),$$

and $\mathcal{L}^1(\mathrm{proj}_{t_i}(R)) \leq \mu$ for each i, we have $|R'| \gtrapprox \mu^4$ where

$$R' = \{(t_1, t_{1'}, t_2, t_{2'}) \in R : |t_1 - t_2| \gtrapprox \mu^2\}. \tag{3.50}$$

Let

$$X = \{(\xi, x, t_0, t_\infty, t_1, t_{1'}, t_2) : (\xi, x) \in F', (t_0, t_\infty) \in P(\xi, x), \text{ and } (t_1, t_{1'}, t_2, w_{t_0,t_\infty}(t_1, t_{1'}, t_2)) \in R'_{t_0,t_\infty}(\xi, x)\}.$$

Integrating everything out, we see that

$$\mathcal{L}^{2(d-1)+5}(X) \gtrapprox \mathfrak{M}(E', F') \left(\inf_{(\xi,x)\in F'} \mu_{\xi,x} \right)^5 \gtrsim \mathcal{L}^{2(d-1)}(F)\lambda^6.$$

For each $k \geq 0$ let

$$X_k = \{(\xi, x, t_0, t_\infty, t_1, t_1', t_2) : \lambda^{C\epsilon}\lambda^{2-k\epsilon} \lesssim |t_1 - t_2| \ll \lambda^{C\epsilon}\lambda^{2-(k+1)\epsilon}\}.$$

Then, recalling that each $\mu_{\xi,x} \gtrsim \lambda$ we have by definition of $R'_{\xi,x}$

$$X = \bigcup_{k \leq \frac{2}{\epsilon}} X_k,$$

and hence for some $X' := X_{k_0}$, we have $\mathcal{L}^{2(d-1)+5}(X') \gtrsim \mathcal{L}^{2(d-1)+5}(X)$. Let $\tilde{\mu}^2 = \lambda^{2-k_0\epsilon}$.

Since each $t_i \in S$, and each $|t_1 - t_2| \approxeq \tilde{\mu}^2$, we see that the $(t_0, t_\infty, t_1, t_{1'}, t_2)$ reside in a set of measure $\lessapprox \mathcal{L}^1(S)^4\tilde{\mu}^2$. Thus we may choose a $(t_0, t_\infty, t_1, t_{1'}, t_2)$ so that, letting

$$F'' = \{(\xi, x) : (\xi, x, t_0, t_\infty, t_1, t_{1'}, t_2) \in X'\},$$

we have

$$\mathcal{L}^{2(d-1)}(F'') \gtrapprox (\mathcal{L}^1(S))^{-4}\tilde{\mu}^{-2}\mathcal{L}^{2(d-1)}(F)\lambda^6. \tag{3.51}$$

We now define G by the condition $G_X = F''$ and recall that

$$|G| = \mathcal{L}^{2(d-1)}(F''), \text{ and } \Omega(G) = \Omega(F'').$$

Let $t_{2'} = w_{t_0,t_\infty}(t_1, t_{1'}, t_2)$. Since $s_{t_0,t_\infty}(t_1, t_{1'}) = s_{t_0,t_\infty}(t_2, t_{2'})$, we may apply Corollary 3.7 to obtain

$$\begin{aligned}\mathcal{L}^{2(d-1)}(F'')^4 &\lesssim \left(D_{t_1,t_2}D_{t_0,t_\infty}D_{t_{1'},t_0}, D_{t'_2,t_0}\right)\Omega(F'') \sup_{t=t_\infty,t_0,t_1,t_{1'},t_2,t_{2'}} \mathcal{L}^{d-1}(\pi_t(G))^7 \\ &\lessapprox \tilde{\mu}^{-2(d-1)}\Omega(F) \sup_{t=t_\infty,t_0,t_1,t_{1'},t_2,t_{2'}} \mathcal{L}^{d-1}(\pi_t(G))^7.\end{aligned} \tag{3.52}$$

By definition of F'', $\pi_{t_i}(G) \subset (E' \cap \gamma_d^{-1}(t_i))$ for $i \in \{0, \infty, 1, 1', 2, 2'\}$. Thus, we may combine (3.51) and (3.52) to obtain

$$\begin{aligned}\mathcal{L}^{2(d-1)}(F)^4\lambda^{24}\tilde{\mu}^{2(d-1)-8}\Omega(F)^{-1} &\lessapprox \mathcal{L}^1(S)^{16} \sup_{t=t_\infty,t_0,t_1,t_{1'},t_2,t_{2'}} \mathcal{L}^{d-1}(E' \cap \gamma_d^{-1}(t))^7 \\ &\lessapprox \mathcal{L}^d(E)^7\mathcal{L}^{2(d-1)}(F)^{-7\epsilon}.\end{aligned} \tag{3.53}$$

Since $\tilde{\mu} \gtrsim \lambda$, we thus have

$$\mathcal{L}^{2(d-1)}(F)^{\frac{4}{7}+C\epsilon}\lambda^{\frac{14+2d}{7}+C\epsilon}\Omega(F)^{-\frac{1}{7}} \lesssim \mathcal{L}^d(E).$$

Since $d \geq 6$, we have $4d + 3 \geq 14 + 2d + C\epsilon$ and hence

$$\left(\mathcal{L}^{2(d-1)}(F)\right)^{\frac{4}{4d+3}+\epsilon}\lambda\Omega(F)^{\frac{3}{4d+3}-\frac{4}{4d+3}} \lesssim \mathcal{L}^d(E)^{\frac{7}{4d+3}}.$$

3.5 Further iteration

By applying Proposition 3.6 once to (3.30), we obtained Corollary 3.7. One obtains the corollary below from $N-1$ iterative applications of Proposition 3.6. This results in an improved value of $\frac{k}{\alpha-\beta}$, but also requires a larger collection of slices which satisfy a more complicated set of conditions. Recall the definition

$$u_{t_0,t_\infty}(t_i,\tilde{s}) = t_0 + \frac{(t_\infty - t_0)(t_i - t_0)}{\tilde{s}(t_i - t_\infty)},$$

and note that $s_{t_0,t_\infty}(t_i, u_{t_0,t_\infty}(t_i,\tilde{s})) = \tilde{s}$, where s_{t_0,t_∞} is as defined in (3.28).

Corollary 3.8. *Let* $N \geq 3$ *and* $\sigma = (t_{0,1}, t_{\infty,1}, s_1, \ldots, t_{0,N-1}, t_{\infty,N-1}, s_{N-1}, t_{0,N}, t_{\infty,N}) \in \mathbb{R}^{3N-1}$. *Let* $\Gamma_{N+1}(\sigma) = \emptyset$, *and for* $1 \leq i \leq N$ *let*

$$\Gamma_i(\sigma) = \{t_{0,i}, t_{\infty,i}\} \cup \Delta_i(\sigma) \cup \Gamma_{i+1}(\sigma)$$

and

$$\Delta_i(\sigma) = \{u_{t_{0,i},t_{\infty,i}}(t, s_i) : t \in \Gamma_{i+1}\}.$$

Suppose that

$$t_{0,i} \notin \Delta_i(\sigma) \cup \Gamma_{i+1}(\sigma) \cup \{t_{\infty,i}\}$$

and

$$t_{\infty,i} \notin \Gamma_{i+1}(\sigma)$$

for each $1 \leq i \leq N$. *Then for any set of lines* G

$$|G|^{\alpha_N} \lesssim \left(\sup_{\tilde{t},\hat{t}} D^{\alpha_N}_{\tilde{t},\hat{t}}\right) \Omega(G)^{\beta_N} \left(\sup_t \mathcal{L}^{d-1}(\pi_t(G))^{k_N}\right) \tag{3.54}$$

where the right sup *ranges over* $t \in \Gamma_1(\sigma)$, *where the left* sup *ranges over* $\tilde{t}, \hat{t}$ *such that*

$$\tilde{t} = t_{0,i},\ \hat{t} \in \{t_{\infty,i}\} \cup \Delta_i(\sigma),\ 1 \leq i \leq N,$$

and where

$$\alpha_1 = 1, \quad \beta_1 = 0, \quad k_1 = 2,$$

and

$$\alpha_{i+1} = 2k_i, \quad \beta_{i+1} = \beta_i + k_i - \alpha_i, \quad \textit{and } k_{i+1} = 4k_i - \alpha_i$$

for $1 \leq i < N$.

One may calculate the formulas

$$k_{i+1} = 4k_i - 2k_{i-1}$$

$$\beta_{i+1} = 2k_{i-1}$$

which give

$$\frac{k_{i+1}}{\alpha_{i+1} - \beta_{i+1}} = 1 + \left(1 - \frac{k_{i-1}}{k_i}\right)^{-1}.$$

Since $\frac{k_i}{k_{i-1}} = 4 - 2\left(\frac{k_{i-1}}{k_{i-2}}\right)^{-1}$, the Banach contraction principle tells us that $\lim_{i\to\infty} \frac{k_i}{k_{i-1}} = 2 + \sqrt{2}$. Thus

$$\lim_{i\to\infty} \frac{k_i}{\alpha_i - \beta_i} = 1 + \sqrt{2}. \tag{3.55}$$

Similarly

$$\lim_{i\to\infty} \frac{k_i}{\alpha_i} = 1 + \frac{\sqrt{2}}{2}. \tag{3.56}$$

In the remainder of this section we use Corollary 3.8 to show that the estimate (3.11) holds with p_N, q_N, r_N satisfying

$$\frac{r_N}{p_N} \geq \frac{k_N}{\alpha_N - \beta_N} - \epsilon, \quad \text{and} \quad \frac{q_N}{p_N} \geq \frac{k_N}{\alpha_N} - \epsilon$$

where ϵ may be taken arbitrarily small. Thus, we obtain Theorem 3.2 from (3.55), (3.56), and Claim 3.4. Due to the complicated nature of the set of slices $\Gamma_1(\sigma)$, we are not able to obtain an appropriately small value of p_N. Hence, for the sake of exposition, we will simplify the argument by passing up several opportunities to slightly improve p_N. For example, we will not employ the two-ends reduction.

Let E', F', λ, S, and $S_{\xi,x}$ be as in Section 3.4. For $(\xi, x) \in F'$ let

$$X_{\xi,x} = \Big\{ \sigma \in \left(S_{\xi,x} \times S_{\xi,x} \times [-C\lambda^{-C_N}, C\lambda^{-C_N}]\right)^{N-1} \times S_{\xi,x} \times S_{\xi,x} \quad \text{such that } \Gamma_1(\sigma) \subset S_{\xi,x}, \text{and such that } |\tilde{t} - \hat{t}| \gtrsim \lambda^{C'_N} \text{ for all } \tilde{t} = t_{0,i},\ \hat{t} \in \Delta_i(\sigma) \cup \Gamma_{i+1}(\sigma) \cup \{t_{\infty,i}\}, \text{and for all } \tilde{t} = t_{\infty,i}, \hat{t} \in \Gamma_{i+1}(\sigma), \text{ where } 1 \leq i \leq N \Big\}, \tag{3.57}$$

where C_N and C'_N will be determined below. One should think of $X_{\xi,x}$ as the set of candidates for σ in Corollary 3.8. Our aim is to find a lower bound for $\mathcal{L}^{3N-1}(X_{\xi,x})$. This will be accomplished by providing the lower bound for a subset Y_N of $X_{\xi,x}$ which is appropriately compatible with the following estimate.

Claim 3.9. *Let $I \subset [0,1]$ with $\mathcal{L}^1(I) < \infty$. For $(t_0, t_\infty, s) \in I \times I \times \mathbb{R}$ let*

$$\widetilde{I}_{t_0,t_\infty,s} = \{t \in I : u_{t_0,t_\infty}(t,s) \in I,\ \text{and } |\tilde{t} - \hat{t}| \gtrsim \mathcal{L}^1(I) \text{ for } \tilde{t} \in \{t, u_{t_0,t_\infty}(t,s)\}, \hat{t} \in \{t_0, t_\infty\}\}. \tag{3.58}$$

Then letting

$$\mathcal{P}(I) = \{(t_0, t_\infty, s) \in I \times I \times [-C\mathcal{L}^1(I)^{-2}, C\mathcal{L}^1(I)^{-2}] : |t_0 - t_\infty| \gtrsim \mathcal{L}^1(I), \quad \text{and } \mathcal{L}^1(\tilde{I}_{t_0,t_\infty,s}) \gtrsim \mathcal{L}^1(I)^6\}, \tag{3.59}$$

we have

$$\int_{\mathcal{P}(I)} \mathcal{L}^1(\widetilde{I}_y)\ dy \gtrsim \mathcal{L}^1(I)^6. \tag{3.60}$$

Proof. We argue as in Section 3.4. Let

$$P = \{(t_0, t_\infty) \in I \times I : |t_0 - t_\infty| \gtrsim \mathcal{L}^1(I)\}$$

and note that

$$\mathcal{L}^2(P) \gtrsim \mathcal{L}^1(I)^2. \tag{3.61}$$

For each $(t_0, t_\infty) \in P$, we let

$$Q_{t_0,t_\infty} = \{(t_1, t_{1'}) \in I \times I : |t_i - t_j| \gtrsim \mathcal{L}^1(I) \text{ for all } i \neq j \in \{0, \infty, 1, 1'\}\}.$$

and note that

$$\mathcal{L}^2(Q_{t_0,t_\infty}) \gtrsim \mathcal{L}^1(I)^2. \tag{3.62}$$

Consider any fixed $t_0, t_\infty \in P$, and let $Q = Q_{t_0,t_\infty}$. Then, as in (3.48), we have

$$\begin{aligned} \mathcal{L}^2(Q) &= \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_Q(t_1, t_{1'})\ dt_1\ dt_{1'} \qquad (3.63) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_Q\left(t_1, u_{t_0,t_\infty}(t_1, s')\right) \left| \frac{(u_{t_0,t_\infty}(t_1, s') - t_0)^2 (t_1 - t_\infty)}{(t_1 - t_0)(t_\infty - t_0)} \right|\ dt_1\ ds' \\ &\lesssim \left(\mathcal{L}^1(I)\right)^{-2} \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_Q\left(t_1, u_{t_0,t_\infty}(t_1, s')\right)\ dt_1\ ds'. \end{aligned}$$

Note that in Q each $|s'| \lesssim \mathcal{L}^1(I)^{-2}$. Thus, letting

$$\overline{S}_{t_0,t_\infty} = \left\{ s' : \int_{\mathbb{R}} \chi_Q\left(t_1, u_{t_0,t_\infty}(t_1, s')\right)\ dt_1 \ll \mathcal{L}^1(I)^6 \right\}$$

we have

$$\int_{\overline{S}_{t_0,t_\infty}} \int_{\mathbb{R}} \chi_Q\left(t_1, u_{t_0,t_\infty}(t_1, s')\right)\ dt_1\ ds' \ll \mathcal{L}^1(I)^4 \lesssim \mathcal{L}^2(Q)\mathcal{L}^1(I)^2. \tag{3.64}$$

Next, we note that if $s \notin \overline{S}_{t_0,t_\infty}$ then $(t_0, t_\infty, s) \in \mathcal{P}(I)$. Thus,

$$\begin{aligned} \int_{\mathcal{P}(I)} \mathcal{L}^1(\tilde{I}_x) dx &\geq \int_P \int_{\mathbb{R} \setminus \overline{S}_{t_0,t_\infty}} \left(\int_{\mathbb{R}} \chi_{Q_{t_0,t_\infty}}\left(t_1, u_{t_0,t_\infty}(t_1, s')\right)\ dt_1 \right)\ ds'\ dt_0\ dt_\infty \\ &\gtrsim \mathcal{L}^1(I)^6, \end{aligned}$$

where the second inequality follows from (3.61), (3.62), (3.63), and (3.64). $\square$

Taking $I = S_{\xi,x}$ in Claim 3.9, we let $Y_1 = \mathcal{P}(S_{\xi,x})$ and for each $(t_0, t_\infty, s) = y \in Y_1$ let $I_y = \widetilde{(S_{\xi,x})}_y$. Recalling that $\mathcal{L}^1(S_{\xi,x}) \gtrsim \lambda$, we have by definition of $\mathcal{P}(S_{\xi,x})$

$$\mathcal{L}^1(I_y) \gtrsim \lambda^6 \text{ for } y \in Y_1 \tag{3.65}$$

and we see from (3.60) that

$$\int_{Y_1} \mathcal{L}^1(I_y)\ dy \gtrsim \lambda^6. \tag{3.66}$$

For $j = 2, \ldots, N-1$, we define Y_j and I_y recursively, letting

$$Y_j = \{(y', y'') : y' \in Y_{j-1} \text{ and } y'' \in \mathcal{P}(I_{y'})\} \subset \mathbb{R}^{3j}$$

and

$$I_{(y',y'')} = \widetilde{(I_{y'})}_{y''} \text{ for } (y', y'') \in Y_j.$$

From (3.65), the definition of $\mathcal{P}(I)$, and induction, we see that

$$\mathcal{L}^1(I_y) \gtrsim \lambda^{6^j} \text{ for } \ y \in Y_j. \tag{3.67}$$

From (3.60), (3.66), (3.67), and induction, we have

$$\begin{aligned} \int_{Y_j} \mathcal{L}^1(I_y)\ dy &= \int_{Y_{j-1}} \int_{\mathcal{P}(I_{y'})} \mathcal{L}^1(\widetilde{(I_{y'})}_{y''})\ dy''\ dy' \\ &\gtrsim \int_{Y_{j-1}} (\mathcal{L}^1(I_{y'}))^6\ dy' \\ &\gtrsim \lambda^{5 \cdot 6^{j-1}} \int_{Y_{j-1}} \mathcal{L}^1(I_{y'})\ dy' \\ &\gtrsim \lambda^{5 \cdot 6^{j-1}} \lambda^{6^{j-1}} = \lambda^{6^j}. \end{aligned} \tag{3.68}$$

Finally, we let

$$Y_N = \{(y, t, t') : y \in Y_{N-1}; t, t' \in I_y;\ \text{and } |t - t'| \gtrsim \mathcal{L}^1(I_y)\}.$$

From (3.67) and (3.68), we have

$$\mathcal{L}^{3N-1}(Y_N) \gtrsim \lambda^{2\cdot 6^{N-1}}.$$

We will now verify that $Y_N \subset X_{\xi,x}$, where in the definition (3.57) of $X_{\xi,x}$ we have $C_N = 2\cdot 6^{N-2}$ and $C'_N = 6^{N-1}$. Let

$$\sigma_N = (t_{0,1}, t_{\infty,1}, s_1, \dots, t_{0,N-1}, t_{\infty,N-1}, s_{N-1}, t_{0,N}, t_{\infty,N}) \in Y_N$$

and for $j = 1, \dots, N-1$ let

$$\sigma_j = (t_{0,1}, t_{\infty,1}, s_1, \dots, t_{0,j}, t_{\infty,j}, s_j),$$

where we note that $\sigma_j \in Y_j$. Additionally, define $I_{\sigma_0} = S_{\xi,x}$. By the definition of Y_i, we have each

$$(t_{0,i}, t_{\infty,i}, s_i) \in \mathcal{P}(I_{\sigma_{i-1}}). \tag{3.69}$$

Since, by (3.67), each $\mathcal{L}^1(I_{\sigma_{i-1}})^{-2} \lesssim \lambda^{-C_N}$, we thus have, by (3.59),

$$\sigma_N \in (S_{\xi,x} \times S_{\xi,x} \times [-C\lambda^{-C_N}, C\lambda^{-C_N}])^{N-1} \times S_{\xi,x} \times S_{\xi,x}.$$

Next, we note that $\Gamma_N(\sigma_N) = \{t_{0,N}, t_{\infty,N}\} \subset I_{\sigma_{N-1}}$ and that, since

$$I_{\sigma_{i-1}} = \widetilde{(I_{\sigma_{i-2}})}_{t_{0,i-1}, t_{\infty,i-1}, s_{i-1}},$$

it follows that

$$\Gamma_i(\sigma_N) \subset I_{\sigma_{i-1}} \Rightarrow \Delta_{i-1}(\sigma_N) \subset I_{\sigma_{i-2}}$$

and thus $\Gamma_{i-1}(\sigma_N) \subset I_{\sigma_{i-2}}$.

So by induction $\Gamma_i(\sigma_N) \subset I_{\sigma_{i-1}}$ for $1 \le i \le N$, and in particular $\Gamma_1(\sigma_N) \subset I_{\sigma_0} = S_{\xi,x}$. Again using (3.69) and (3.59), we obtain

$$|t_{0,i} - t_{\infty,i}| \gtrsim \mathcal{L}^1(I_{\sigma_{i-1}}) \gtrsim \lambda^{C'_N}.$$

Also, since $\Gamma_{i+1}(\sigma_N) \subset I_{\sigma_i} = \widetilde{(I_{\sigma_{i-1}})}_{t_{0,i},t_{\infty,i},s_i}$, we see that

$$|\tilde{t} - \hat{t}| \gtrsim \mathcal{L}^1(I_{\sigma_{i-1}}) \gtrsim \lambda^{C'_N} \text{ for } \tilde{t} \in \{t_{0,i}, t_{\infty,i}\},\ \hat{t} \in \Delta_i(\sigma_N) \cup \Gamma_{i+1}(\sigma_N).$$

Thus $\sigma_N \in X_{\xi,x}$ and $Y_N \subset X_{\xi,x}$. In particular

$$\mathcal{L}^{3N-1}(X_{\xi,x}) \gtrsim \lambda^{2\cdot 6^{N-1}}.$$

Let

$$X = \{(\xi, x, \sigma) : (\xi, x) \in F' \text{ and } \sigma \in X_{\xi,x}\}$$

and note that

$$\mathcal{L}^{2(d-1)+3N-1}(X) \gtrsim \lambda^{2\cdot 6^{N-1}} \mathcal{L}^{2(d-1)}(F').$$

Since the σ's reside in a set of measure $\lesssim \mathcal{L}^1(S)^{2N} \lambda^{-(N-1)C_N}$, we see that, letting $C''_N = 2 \cdot 6^{N-1} + (N-1)C_N$, we may find a fixed σ so that

$$\mathcal{L}^{2(d-1)}(F'') \gtrsim \lambda^{C''_N} \mathcal{L}^1(S)^{-2N} \mathcal{L}^{2(d-1)}(F') \tag{3.70}$$

where

$$F'' = \{(\xi, x) : (\xi, x, \sigma) \in X\}.$$

Since $\mathfrak{M}(E, F') \gtrsim \lambda \mathcal{L}^{2(d-1)}(F)$ and $T[\chi_E] \leq 1$, we have

$$\mathcal{L}^{2(d-1)}(F') \gtrsim \lambda \mathcal{L}^{2(d-1)}(F). \tag{3.71}$$

We now apply Corollary 3.8 with the set of lines G defined by $G_X = F''$. We then have

$$|G| = \mathcal{L}^{2(d-1)}(F'')$$

and

$$\Omega(G) = \Omega(F'') \leq \Omega(F).$$

Also, since $\Gamma_1(\sigma) \in S_{\xi,x}$ for every $(\xi, x) \in F''$, we have $\pi_t(G) \subset E \cap \gamma_d^{-1}(S)$ for every $t \in \Gamma_1(\sigma)$. Furthermore, by definition of $X_{\xi,x}$, we have

$$\sup_{\tilde{t},\hat{t}} D_{\tilde{t},\hat{t}} \lesssim \lambda^{-(d-1)C'_N}.$$

where the sup ranges over

$$\tilde{t} = t_{0,i},\ \hat{t} \in \{t_{\infty,i}\} \cup \Delta_i(\sigma),\ 1 \leq i \leq N.$$

Thus, from (3.54) we obtain

$$\mathcal{L}^{2(d-1)}(F'')^{\alpha_N} \lesssim \left(\lambda^{-(d-1)C'_N}\right)^{\alpha_N} \Omega(F)^{\beta_N} \left(\sup_{t \in S} \mathcal{L}^{d-1}(E \cap \gamma_d^{-1}(t))^{k_N}\right). \tag{3.72}$$

Noting that $\frac{k_N}{\alpha_N} < 2$ (and certainly $< 2N$) and

$$\mathcal{L}^1(S) \left(\sup_{t \in S} \mathcal{L}^{d-1}(E) \cap \gamma_d^{-1}(t)\right) \lesssim \lambda^{-\epsilon} \mathcal{L}^{2(d-1)}(F)^{-\epsilon} \mathcal{L}^d(E),$$

we obtain from (3.70), (3.71), and (3.72)

$$\mathcal{L}^{2(d-1)}(F)^{\alpha_N + k_N\epsilon} \lambda^{C'''_N \alpha_N + k_N \epsilon} \Omega(F)^{-\beta_N} \lesssim \mathcal{L}^d(E)^{k_N}.$$

where $C'''_N = 1 + (d-1)C'_N + C''_N$.

Finally, this gives

$$\lambda \mathcal{L}^{2(d-1)}(F)^{\frac{1}{q_N}} \Omega(F)^{\left(\frac{1}{q_N} - \frac{1}{r_N}\right)} \lesssim \mathcal{L}^d(E)^{\frac{1}{p_N}}$$

where

$$\begin{aligned} p_N &= \frac{C'''_N \alpha_N + k_N \epsilon}{k_N} \geq 6^{N-2}(6(d+1) + 2(N-1))\frac{\alpha_N}{k_N} + \epsilon \\ q_N &= \frac{C'''_N \alpha_N + k_N \epsilon}{\alpha_N + k_N \epsilon} \geq \left(\frac{k_N}{\alpha_N} - C\epsilon\right) p_N \\ r_N &= \frac{C'''_N \alpha_N + k_N \epsilon}{\alpha_N - \beta_N + k_N \epsilon} \geq \left(\frac{k_N}{\alpha_N - \beta_N} - C\epsilon\right) p_N. \end{aligned} \tag{3.73}$$

Thus, from (3.55), we see that by taking N large and ϵ small, we have $\frac{r}{p}$ arbitrarily close to $1+\sqrt{2}$ (although this comes with the price of a very large p).

3.6 Nikodym mixed-norms

We now follow a method of Tao from [40] to show that Corollary 3.3 follows from Theorem 3.1. For $z \in \mathbb{R}^d$ write $x_z = \text{proj}_H(z)$ and $t_z e_d = \text{proj}_{e_d}(z)$ where proj denotes orthogonal projection. For nonnegative integers j let

$$S_j = \{z \in \mathbb{R}^d : 2^{-(j+1)} < t_z \leq 2^{-j}\}.$$

We first prove Corollary 3.3 in the special case when f is supported on S_0. In fact, to prove this case it suffices, since T is local and $p \leq q \leq r$, to consider the case when f is supported on $S_0 \cap Q$ where Q is the cube centered at $\frac{1}{2}e_d$ with side length 1. Furthermore, assume that f is positive.

We then consider the projective transformation

$$\phi(z) = \frac{x_z + e_d}{t_z}.$$

The idea is that we have $T[f](\xi, x) \approx T[f \circ \phi](x, \xi)$. To line everything up, we will in fact estimate $T[f \circ \phi \circ d_2]$ where $d_2(y) = 2y$. Then

$$\begin{aligned} T[f \circ \phi \circ d_2](\xi, x) &= \int_{\frac{1}{2}}^{1} f \circ \phi \circ d_2(x + t(\xi + e_d))\, dt \\ &= \int_{\frac{1}{2}}^{1} f\left(\xi + \frac{1}{2t}(2x + e_d)\right)\, dt \\ &\approx \int_{\frac{1}{2}}^{1} f(\xi + \tilde{t}(2x + e_d))\, d\tilde{t} \\ &= T[f](2x, \xi), \end{aligned}$$

where the $\approx$ follows from the fact that the Jacobian $\frac{1}{2t^2} \approx 1$ on $[\frac{1}{2}, 1]$ and where the first and last equations follow from the fact that f is supported on S_0. Thus by Theorem 3.1

$$\begin{aligned}
&\left(\int_{\mathbb{R}^{d-1}} \left(\int_{B(0,C)} |T[f](\xi, x)|^r \, d\xi\right)^{\frac{q}{r}} \, dx\right)^{\frac{1}{q}} \\
&= \left(\int_{B(0,C')} \left(\int_{B(0,C)} |T[f](\xi, x)|^r \, d\xi\right)^{\frac{q}{r}} \, dx\right)^{\frac{1}{q}} \\
&\lesssim \left(\int_{B(0,C')} \left(\int_{B(0,C)} \left|T[f \circ \phi \circ d_2]\left(x, \frac{1}{2}\xi\right)\right|^r \, d\xi\right)^{\frac{q}{r}} \, dx\right)^{\frac{1}{q}} \\
&\lesssim \left(\int_{B(0,C')} \left(\int_{\mathbb{R}^{d-1}} |T[f \circ \phi \circ d_2](x, \xi)|^r \, d\xi\right)^{\frac{q}{r}} \, dx\right)^{\frac{1}{q}} \\
&\lesssim \|f \circ \phi \circ d_2\|_{L^p(\mathbb{R}^d)}.
\end{aligned}$$

where we use the fact that f is supported on Q for the first equation. Since f is supported on S_0, we have

$$\|f \circ \phi \circ d_2\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}.$$

Lifting our support assumptions on f, we note that

$$\|T[f]\|_{L^q(L^r),N} \leq \sum_{j=0}^{\infty} \|T[\chi_{S_j} f]\|_{L^q(L^r),N}.$$

However each $\chi_{S_j} f \circ d_{2^{-j}}$ is supported on S_0, and

$$T[\chi_{S_j} f](\xi, x) = 2^{-j} T[\chi_{S_j} f \circ d_{2^{-j}}](\xi, 2^j x).$$

Thus, for each j

$$\begin{aligned}
\|T[\chi_{S_j} f]\|_{L^q(L^r),N} &= 2^{-j(1+\frac{d-1}{q})} \|T[\chi_{S_j} f \circ d_{2^{-j}}]\|_{L^q(L^r),N} \\
&\lesssim 2^{-j(1+\frac{d-1}{q})} \|\chi_{S_j} f \circ d_{2^{-j}}\|_{L^p(\mathbb{R}^d)} \\
&\leq 2^{-j(1+\frac{d-1}{q}-\frac{d}{p})} \|f\|_{L^p(\mathbb{R}^d)}.
\end{aligned}$$

Hence, provided (3.8) holds with strict inequality, which is indeed the case under the assumptions of Corollary 3.3, we have

$$\sum_{j=0}^{\infty} \|T[\chi_{S_j} f]\|_{L^q(L^r),N} \lesssim \|f\|_{L^p(\mathbb{R}^d)}.$$

Chapter 4

Finite fields

When estimating the size of Kakeya sets in $\mathbb{R}^d$, we are interested in lower bounds for the Hausdorff or Minkowski dimension of the union of a direction-separated collection of lines. Due to the nature of the quantities in question, it is often desirable to replace the collection of lines by a δ-separated collection of δ-neighborhoods of lines, in effect discretizing our Kakeya set to scale δ. Unfortunately this discretization process obscures some of the structure which should be present in E. For example, while two distinct lines intersect in at most 1 point, the discretization of two δ separated lines can intersect in many discretized points. Similar "small-angle" considerations apply to (d, k) sets in general. The difficulty of overcoming this and related issues can tend to hide the true nature of the problem, and for this reason it is instructive to replace Euclidean space by vector spaces over finite fields.

In the sense that it retains access to linear algebra, a finite field F is an ideal model for the discretization of the unit interval in $\mathbb{R}$ to scale $\frac{1}{|F|}$. In this setting, a (d, k) set is defined to be a subset of F^d containing a translate of every k-dimensional plane in F^d. For a given field F, there is not a suitable analogue of Hausdorff or Minkowski dimension. Instead, we are interested in uniform estimates over large families of F. We let $\mathrm{ffdim}(d, k, p)$ be the supremum of the set of values η such that there exist $C_{d,k,p,\eta} > 0$

with

$$|E| \geq C_{d,k,p,\eta}|F|^{\eta} \tag{4.1}$$

for all F with characteristic at least p and all (d,k) sets E in F^d. The statement that $\text{ffdim}(d,k,p) \geq \beta$ is then analogous to the statement that (d,k) sets in $\mathbb{R}^d$ have Minkowski dimension at least β. Ideally we would like to prove such statements with $p = 2$, but the main goal is to improve β regardless of p.

The primary drawback of working in the setting of finite fields is that it is difficult to model Euclidean objects with size between $\frac{1}{|F|}$ and 1, such as a cube of side-length ω where $\frac{1}{|F|} \ll \omega \ll 1$. Thus, for instance, it does not seem to be possible to use induction on scales. This difficulty can be overcome in part by restricting ourselves to the fields $\mathbb{Z}_p$, where the notion of order in $\mathbb{R}$ is locally retained. Indeed, this would seem to be a natural restriction since any finite field is a field-extension of some $\mathbb{Z}_p$, and in the Euclidean setting we generally restrict ourselves to considering the Kakeya problem over $\mathbb{R}$ rather than its field-extension $\mathbb{C}$. However, even with this sort of restriction, the model of the ω-cube would not completely retain the structure of F^d. This reflects the lack of a dilation symmetry in discrete settings.

4.1 A recursive estimate for (d,k) sets

In Chapters 2 and 3, we derive dimension estimates for (d,k) sets in $\mathbb{R}^d$ as a corollary to certain maximal operator bounds for k-planes. These bounds are obtained, through an iterative process, from maximal operator bounds and mixed-norm estimates for lines. It is likely that similar operator estimates hold in the setting of finite fields, but in the interest of keeping this chapter non-technical we prove weaker estimates of the form (4.3) which are still strong enough to obtain the desired dimension estimates. In any case, the iterative process in Chapter 2 does not generally provide "sharp p,q" type bounds. For known sharp-type k-plane maximal operator bounds in F^d, see [9].

The proposition below is a recursive dimension estimate, analogous to the recursive maximal operator/mixed-norm estimate from Chapter 2.

Proposition 4.1. *Let F be a finite field, and $1 \leq k < d-1$. Suppose that for every (d,k) set E,*

$$|E| \gtrsim |F|^{\eta} \tag{4.2}$$

and suppose that for every set L of lines in F^{d+1}, no more than M of which are parallel to each direction,

$$\left|\bigcup_{l \in L} l\right| \gtrsim M^{-\alpha}|L|^{\beta}|F|^{\gamma}. \tag{4.3}$$

Then for every $(d+1,k+1)$ set E,

$$|E| \gtrsim |F|^{\eta(\beta-\alpha)+d\beta+\gamma}. \tag{4.4}$$

The X-ray estimates which we will prove in Propositions 4.5 and 4.8 have the value $\gamma = 1$. In this case, by taking L to be the collection of all lines, one may check that the sharp exponent of M, which we obtain, is $\alpha = 2\beta - 1$. With these exponents, we may rewrite (4.4) in the more pleasing form

$$|E| \gtrsim |F|^{(\eta+1)(1-\beta)+(d+1)\beta}.$$

Combining Propositions 4.1, 4.5, and 4.7, we obtain

Theorem 4.2.

$$\text{ffdim}(d,k,4) \geq \max\left(d - \left(\frac{3}{7}\right)^{k}(d-k),\ d - \left(\frac{3}{7}\right)^{k-1}\frac{d-k-1}{2}\right).$$

Previously, the best known dimension estimate was $d - \frac{d-k-1}{k+1}$, from [9]. The left hand side of the max is obtained by using Proposition 4.5 for both dimension and X-ray estimates. The right hand side of the max is superior when $d-(k-1) < 8$ and is obtained by instead starting with the dimension estimate from Proposition 4.7. Further improvement is possible when $d-(k-1) < 6$ by using the X-ray estimate from Proposition 4.8. By using more advanced arithmetic-combinatorial X-ray estimates such as the finite field analog of the Theorem 3.2, it is likely that $\frac{7}{3}$ could be replaced in all dimensions by numbers arbitrarily close to $1+\sqrt{2}$ (which is still inferior to the value $\frac{5}{2}$ from Proposition 4.8) by taking p sufficiently large. Also, note that we do not use an analogue of the L^2 method from Chapter 2, since the corresponding finite field estimates do not seem to be known at present. Finally, readers interested in field-extensions of $\mathbb{Z}_2$ and $\mathbb{Z}_3$ should replace $\frac{7}{3}$ with 2.

Proof of Proposition 4.1. Let $E \subset F^{d+1}$ be a $(d+1, k+1)$ set. Let $e_1, \ldots, e_{d+1}$ be the standard basis for F^{d+1} and H denote the hyperplane $\text{span}(e_1, \ldots, e_d)$. For each $\xi \in H$ let E_ξ be the set of points $x \in H$ so that the line $l(x, x+e_{d+1}+\xi)$ determined by x and $x+e_{d+1}+\xi$ is contained in E.

We want to observe that for each ξ, E_ξ is a (d,k) set. Let P be a k-dimensional subspace of $F^d = H$. Then, since E is a $(d+1,k+1)$ set, there exists an $x_0 \in F^{d+1}$ so that $x_0 + \text{span}(\xi + e_{d+1}, P)$ is contained in E. Possibly subtracting a multiple of $(\xi + e_{d+1})$, we may assume that $x_0 \in H$. Then the k-plane $x_0 + P$ is contained in E_ξ. Since P was arbitrary, we see that E_ξ is a (d,k) set.

From the hypothesized bound (4.2), we thus have

$$|E_\xi| \gtrsim |F|^\eta$$

for every ξ. Choose each $\tilde{E}_\xi \subset E_\xi$ so that

$$|\tilde{E}_\xi| \approx |F|^\eta. \tag{4.5}$$

Define the collection of lines in F^{d+1}

$$L = \{l(x, x + e_{d+1} + \xi) : x \in \tilde{E}_\xi, \xi \in H\}.$$

Since $|H| = F^d$, we have $|L| \approx |F|^{\eta+d}$. By definition of E_ξ,

$$\bigcup_{l \in L} l \subset E.$$

The classes of parallel lines in L are the sets $L_\xi = \{l(x, x + e_{d+1} + \xi) : x \in \tilde{E}_\xi\}$, so from (4.5) we see that no more than $\approx |F|^\eta$ lines in L are parallel. It follows from (4.3) that

$$\begin{aligned} |E| \geq \left|\bigcup_{l \in L} l\right| &\gtrsim (|F|^\eta)^{-\alpha} \left(|F|^{\eta+d}\right)^\beta |F|^\gamma \\ &= |F|^{\eta(\beta-\alpha)+d\beta+\gamma}. \end{aligned}$$

□

4.2 The bush and Cordoba arguments

We now proceed to the task of proving estimates suitable for use with Proposition 4.1. In this section, we give two bounds which are, for the most part, weaker than those given in later sections, but which serve to illustrate the technical simplifications afforded by the setting of finite-fields.

First, we prove an estimate which would be implied by the analogue of Drury and Christ's $L^{\frac{d+1}{2}} \to L^{d+1}(L^{d+1})$ X-ray transform bound. We use Bourgain's bush argument; this is one of a multitude of proofs of (4.6), also see [29].

Proposition 4.3. *Let F be a finite field, and let $d \geq 2$. For every collection L of lines in F^d*

$$\left|\bigcup_{l \in L} l\right| \gtrsim |L|^{\frac{1}{2}}|F|. \tag{4.6}$$

Proof. Let $E = \bigcup_{l \in L} l$ and $\mu = \sup_{x \in E} |\{l \in L : x \in l\}|$. We observe two lower bounds for $|E|$; the first bound is favorable for small μ and the second bound is favorable for large μ.

Let $I = \{(x, l) : x \in l, l \in L\}$. Then

$$\begin{aligned} |E| \geq \frac{1}{\mu}\sum_{x \in E} |\{l \in L : x \in l\}| = \frac{1}{\mu}\sum_{x \in E}\sum_{l \in L} \chi_I(x, l) &= \frac{1}{\mu}\sum_{l \in L}\sum_{x \in E} \chi_I(x, l) \\ &= \frac{1}{\mu}|L||F|. \end{aligned} \tag{4.7}$$

On the other hand, let x_0 satisfy $|\{l \in L : x_0 \in l\}| = \mu$. We form the "bush"

$$B = \bigcup_{l \in L : x_0 \in L} l.$$

Two distinct lines intersect in at most one point, and thus the lines in B are disjoint away from x_0. This implies that

$$|E| \geq |B| \geq \mu(|F| - 1) \gtrsim \mu|F|. \tag{4.8}$$

We finish by taking the geometric mean of (4.7) and (4.8). □

Next, we have an estimate in the spirit of Cordoba's bound [11]. Our bound implies that ffdim$(d, d-1, 2) = d$ for $d \geq 2$.

Proposition 4.4. *Let F be a finite field, and let $1 \leq k < d$. For every collection L of k-planes in F^d such that $|L| \leq |F|$*

$$\left|\bigcup_{l \in L} l\right| \gtrsim |L||F|^k.$$

Proof. Let $E = \bigcup_{l \in L} l$, and $I = \{(x, l) : x \in l, l \in L\}$. For each $x \in E$ let

$$\mu(x) = |\{l \in L : x \in l\}|.$$

Then for every $l \in L$

$$\sum_{x \in l} \mu(x) = \sum_{x \in l} \sum_{l' \in L} \chi_I(x, l') = |F|^k + \sum_{x \in l} \sum_{l' \neq l \in L} \chi_I(x, l')$$
$$= |F|^k + \sum_{l' \neq l \in L} \sum_{x \in l} \chi_I(x, l').$$

For any $l' \neq l$, we have $|l' \cap l| \leq |F|^{k-1}$. By hypothesis $|L \setminus \{l\}| \leq |F|$, and thus

$$\sum_{l' \neq l \in L} \sum_{x \in l} \chi_I(x, l') \leq |F||F|^{k-1}$$

giving

$$\sum_{x \in l} \mu(x) \leq 2|F|^k.$$

For each l, we thus have $|\{x \in l : \mu(x) \geq 4\}| \leq \frac{|F|^k}{2}$ and hence

$$|\{x \in l : \mu(x) \leq 4\}| \geq \frac{|F|^k}{2}.$$

Letting $I' = \{(x, l) : x \in l, l \in L, \mu(x) \leq 4\}$, we thus have $|I'| \geq |L| \frac{|F|^k}{2}$. Thus, as in (4.7), we have

$$|E| \geq \frac{1}{8}|L||F|^k.$$

□

4.3 The $\frac{7}{3}$ X-ray estimate.

The following proposition would follow from an $L^{\frac{4d+3}{7}} \to L^{\frac{4d+3}{4}}(L^{\frac{4d+3}{3}})$ mixed-norm estimate for the X-ray transform in F^d.

Proposition 4.5. *Let F be a finite field with characteristic strictly greater than 3, and let $d \geq 2$. For every collection L of lines in F^d so that at most M lines in L are parallel to each direction*

$$\left|\bigcup_{l \in L} l\right| \gtrsim M^{-\frac{1}{7}} |L|^{\frac{4}{7}} |F|. \tag{4.9}$$

To obtain a restricted weak-type version of the corresponding X-ray transform bound, one would additionally need to consider unions of collections of λ-density subsets of lines, where $\lambda < 1$.

By considering the set of all lines, where $|L| = |F|^{2(d-1)}$ and $M = |F|^{d-1}$, one sees that the exponent $-\frac{1}{7}$ for M is sharp for the given exponents of $|L|$ and $|F|$.

Before proving Proposition 4.5, we state the ubiquitous "combinatorial Cauchy-Schwarz" inequality.

Lemma 4.6. *Suppose A and B are finite sets, and $I \subset A \times B$. Then*

$$|\{(a, b, b') : (a, b), (a, b') \in I\}| \geq \frac{|I|^2}{|A|}.$$

Proof. We note that

$$|\{(a, b, b') : (a, b), (a, b') \in I\}| = \sum_{a \in A} \left(\sum_{b \in B} \chi_I(a, b) \right)^2,$$

and

$$|I| = \sum_{a \in A} \sum_{b \in B} \chi_I(a, b).$$

Applying Cauchy-Schwarz,

$$\sum_{a \in A} \sum_{b \in B} \chi_I(a, b) \left(\leq \sum_{a \in A} \left(\sum_{b \in B} \chi_I(a, b) \right)^2 \right)^{\frac{1}{2}} |A|^{\frac{1}{2}}.$$

□

Alternatively, if one is willing to lose a factor of 4 (which is permissible for every application we consider), Lemma 4.6 can be proven by observing that for "the average a" there are $\frac{|I|}{|A|}$ points $b \in B$ such that $(a, b) \in I$.

Second Proof. Let I_p be the set of "popular" incidences

$$I_p = \left\{(a, b) \in I : |\{(a, b') \in I : b' \in B\}| \geq \frac{|I|}{2|A|}\right\}.$$

By design, $|I \setminus I_p| \leq \frac{|I|}{2}$ and so $|I_p| \geq \frac{|I|}{2}$. Letting

$$V_p = \{(a, b, b') : (a, b), (a, b') \in I_p\},$$

we clearly have

$$\frac{|I|^2}{4|A|} \leq |I_p|\frac{|I|}{2|A|} \leq |V_p| \leq |\{(a, b, b') : (a, b), (a, b') \in I\}|.$$

□

Proof of Proposition 4.5. We follow an argument in [29], but we do not assume that our collection of lines is direction-separated. Let E denote $\cup_{l \in L} l$. Consider the set of double point line incidences

$$I' = \{(x, y, l) : x, y \in l; l \in L; x \neq y\},$$

the set of "double pointed angles"

$$V'' = \{((x_1, y_1, l_1), (x_2, y_2, l_2)) \in I'^2 : x_1 = x_2\},$$

and the set of quadrilaterals

$$Q = \{(((x_{1,1}, y_{1,1}, l_{1,1}), (x_{1,2}, y_{1,2}, l_{1,2})), ((x_{2,1}, y_{2,1}, l_{2,1}), (x_{2,2}, y_{2,2}, l_{2,2}))) \in V''^2 : y_{1,1} = y_{2,1}, y_{1,2} = y_{2,2}\}. \quad (4.10)$$

Above, we allow for degenerate configurations such as angles v with $l_1 = l_2$. Due to "correct parity" this does not seem to affect the argument, however if one desired these degeneracies could be easily eliminated.

We obtain (4.9) by giving upper and lower bounds for the number of quadrilaterals, $|Q|$. Since $|F| \geq 2$ and each line contains $|F|$ points

$$|I'| \approx |F|^2|L|. \tag{4.11}$$

By Lemma 4.6,

$$|V''| \gtrsim \frac{|I'|^2}{|E|}. \tag{4.12}$$

Finally, another application of Lemma 4.6 gives

$$|Q| \gtrsim \frac{|V''|^2}{|E|^2}. \tag{4.13}$$

We now proceed to the upper bound for $|Q|$. For $x \neq y \in F^d$, let $l(x, y)$ denote the line determined by x and y. It will be convenient to use the definition, equivalent to (4.10),

$$Q = \left\{(x_0, x_1, x_2, x_3) \in E^4 : x_i \neq x_{i+1},\ l(x_i, x_{i+1}) \in L,\ \text{for } i \in \{0, 1, 2, 3\}\right\}$$

where above we adopt the convention that $x_4 := x_0$. Our upper bound follows the heuristic "a quadrilateral is determined, up to a factor of $M|F|$, by three points." This is analogous to the fact that a line is determined by two points. Define the projections from Q to F^d

$$\pi_0(q) := \frac{x_0 + x_1}{2},$$

$$\pi_1(q) = \frac{x_1 + 2x_2}{3},$$

$$\pi_2(q) = 2x_2 - x_3.$$

Noting that each π_i maps into the line $l(x_i, x_{i+1})$ and hence into E, we obtain the bounds

$$|\pi_i(Q)| \leq E \tag{4.14}$$

for $i = 0, 1, 2$. Since

$$x_0 - x_3 = 2\pi_0 - 3\pi_1 + \pi_2,$$

we see that the direction of the line $l(x_0, x_3)$ is determined by (π_0, π_1, π_2). By assumption there are at most M lines from L in a given direction, and for each such line there are at most $|F|$ possible values of x_0.

However, once x_0, $l(x_0, x_3)$, and π_0, π_1, π_2 are known we can fully reconstruct q. Thus, there are at most $M|F|$ quadrilaterals in Q which share values of π_0, π_1, π_2 and so from (4.14) we obtain the upper bound

$$|Q| \leq M|F||E|^3.$$

Combining this with (4.11), (4.12), and (4.13), we obtain (4.9). $\square$

From the case $M = 1$ of Proposition 4.5, we see, as in [29], that $\mathrm{ffdim}(d, 1, 4) \geq \frac{4d+3}{7}$.

4.4 Wolff's estimate for $(d, 1)$ sets.

For $d < 8$, the dimension estimate for Kakeya sets implied by Proposition 4.5 is superceded by Wolff's estimate. Although Wolff's original "hairbrush" argument still applies in the setting of finite fields (see [45]), we instead follow the triangle-counting argument of Katz from [29].

Proposition 4.7. *Let F be a finite field, and let L be a collection of direction separated lines in F^d so that $|L| \gtrsim |F|$. Then*

$$\left|\bigcup_{l\in L}\right| \gtrsim |L|^{\frac{1}{2}}|F|^{\frac{3}{2}}. \tag{4.15}$$

Proof. Let E denote $\cup_{l\in L} l$. We consider the set of point line incidences

$$I = \{(x,l) : x \in l; l \in L\},$$

the set of non-degenerate angles

$$V = \{((x_1,l_1),(x_2,l_2)) \in I^2 : x_1 = x_2, l_1 \neq l_2\},$$

the set of "single pointed angles"

$$V' = \{(y,(x_1,l_1),(x_2,l_2)) \in E \times V : y \in l_1, y \neq x_1\},$$

and the set of non-degenerate triangles

$$T = \{((y_1,(x_{1,1},l_{1,1}),(x_{1,2},l_{1,2}))\,(y_2,(x_{2,1},l_{2,1}),(x_{2,2},l_{2,2}))) \in V'^2 : \\ y_1 = y_2, l_{1,2} = l_{2,2}, l_{1,1} \neq l_{2,1}\}.$$

We obtain (4.15) by giving lower and upper bounds for $|T|$. Starting with the lower bound, we note that since each line contains $|F|$ points

$$|I| \geq |L||F|.$$

By Lemma 4.6,

$$|\tilde{V}| \gtrsim \frac{|I|^2}{|E|} \tag{4.16}$$

where $\tilde{V}$ is the set of possibly degenerate angles. Since the set of degenerate angles, where $l_1 = l_2$, is in correspondence with I, we see that the lower bound

$$|V| \gtrsim \frac{|I|^2}{|E|} \tag{4.17}$$

would follow if $|\tilde{V}| \gg I$. We guarantee this condition by assuming that $|E| \ll |I|$, which is without loss of generality since $|L| \gtrsim |F|$.

Since $|F| \geq 2$,

$$|V'| \gtrsim |F||V|.$$

Finally, applying Lemma 4.6 again, we obtain

$$|T| \gtrsim \frac{|V'|^2}{|E||L|}. \tag{4.18}$$

To guarantee (4.18), we must assume without loss of generality that $|E| \ll |L|^{\frac{1}{2}}|F|^{\frac{3}{2}}$, implying $\frac{|V'|}{|E||L|} \gg 1$ and thus that the number of triangles is much greater than the number of degenerate triangles, where $l_{1,1} = l_{2,1}$.

To obtain the upper bound for $|T|$, we note that for a fixed angle v the number of triangles $((y_1, v), (y_2, v'))$ is bounded above by the number of lines $l_{2,1}$ in L co-planar to v. Since there are $|F|$ directions co-planar to each v and each line in L points in a different direction, we obtain

$$|T| \leq |V||F|. \tag{4.19}$$

Combining this upper bound with (4.18) yields (4.15). It is worth noting that the upper bound (4.19) required nondegeneracy only of the angle collection V and not the triangle collection T. In the next section, we will give a different upper bound which will require the nondegeneracy of T. □

From Proposition 4.7, we see that $\text{ffdim}(d, 1, 2) \geq \frac{d+2}{2}$.

4.5 An improved X-ray estimate in dimensions 3-6.

In Proposition 4.7, we only considered the case of a direction separated collection of lines. Following the proof and allowing for at most M parallel lines, one obtains the bound

$$\left|\bigcup_{l\in L}\right| \gtrsim M^{-\frac{1}{2}}|L|^{\frac{1}{2}}|F|^{\frac{3}{2}}.$$

However, since at most M lines are parallel, we could have applied Proposition 4.7 to a direction separated subcollection $L' \subset L$ of lines with $|L'| \geq \frac{|L|}{M}$ to obtain the same result. In fact, after considering the example where L is the collection of all lines, we conjecture that the sharp exponent of M should be $-\frac{1}{2(d-1)}$.

In Euclidean space, this sharp estimate was shown to hold by Wolff when $d = 3$. Wolff's argument was generalized to $d \geq 4$ by Łaba and Tao to obtain an interpolant of the conjectured sharp estimate. The argument used by Wolff, Łaba, and Tao, involves objects (slabs) of intermediate scale, and thus does not seem to immediately transfer to the setting of finite fields (except possibly $\mathbb{Z}_p$).

By using an "arithmetic" upper bound for triangles, we obtain an improvement to Proposition 4.5 when $d \leq 6$.

Proposition 4.8. *Let F be a finite field with characteristic strictly greater than 2. Suppose $d \leq 6$ and let L be a collection of lines in F^d so that at most M lines in L are parallel to each direction. Then*

$$\left|\bigcup_{l\in L} l\right| \gtrsim M^{-\frac{1}{5}}|L|^{\frac{3}{5}}|F|. \tag{4.20}$$

As in Proposition 4.5, the exponent of M is sharp. In terms of dimension estimates for Kakeya sets, the bound (4.20) matches Wolff's estimate when $d = 6$, and is worse when $d < 6$. The mixed norm estimate corresponding to (4.20) would be an $L^{\frac{3d+2}{5}} \to L^{\frac{3d+2}{3}}\left(L^{\frac{3d+2}{2}}\right)$ bound, which in Euclidean space is not implied by Łaba and Tao's bound.

Proof of Proposition 4.8. Recall from the proof of Proposition 4.7 the definitions of E, I, V, V', T, and the lower bound

$$|T| \gtrsim \frac{|V'|^2}{|E||L|} \gtrsim \frac{|L|^3|F|^6}{|E|^3}. \tag{4.21}$$

To ensure the non-degeneracy of angles, we made the assumption $|E| \ll |I|$, which we may repeat here (in fact without the additional hypothesis necessary for Proposition 4.7). To ensure the non-degeneracy of triangles, we assumed

$$|E| \ll |L|^{\frac{1}{2}}|F|^{\frac{3}{2}}. \tag{4.22}$$

To have (4.20) implied by the negation of (4.22), we need

$$M^{-\frac{1}{5}}|L|^{\frac{3}{5}}|F| \lesssim |L|^{\frac{1}{2}}|F|^{\frac{3}{2}}.$$

Since $|L| \leq M|F|^{d-1}$, this is guaranteed when $d \leq 6$.

We now mimic the upper bound for quadrilaterals from Proposition 4.5. Rewrite

$$T = \Big\{(x_0, x_1, x_2) \in E^3 : x_i \neq x_j, l(x_i, x_{i+1}) \in L, l(x_i, x_{i+1}) \neq l(x_j, x_{j+1}) \text{ for } i \neq j \in \{0, 1, 2\}\Big\} \tag{4.23}$$

where above we adopt the convention that $x_3 := x_0$. Define the projections from $T \to E$

$$\pi_0(t) = \frac{x_0 + x_1}{2}$$

and

$$\pi_1(t) = \frac{x_1 + x_2}{2}.$$

Then $x_0 - x_2 = 2(\pi_0 - \pi_1)$, and thus the direction of $l(x_0, x_2)$ is determined by (π_0, π_1). As in Proposition 4.5, we conclude that

$$|T| \leq M|F||E|^2$$

which, combined with (4.21), yields (4.20).

□

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