ESTIMATES FOR THE X-RAY TRANSFORM RESTRICTED TO 2-MANIFOLDS

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ABSTRACT. We prove almost sharp mixed-norm estimates for the X-ray transform restricted to lines whose directions lie on certain well-curved two dimensional manifolds.

1. INTRODUCTION

The full X-ray transform, also known as the 1-plane transform, X_{full}, is an operator from the functions on $\mathbb{R}^d$ to the functions on $\mathcal{G}_d$, the space of all lines in $\mathbb{R}^d$. It is defined as

$$X_{full}f(l) = \int_l f, \quad l \in \mathcal{G}_d.$$

It is well-known [2, 20, 14] that the optimal conjectured mixed-norm estimates for X_{full} imply the Kakeya conjecture, which states that every compact subset of $\mathbb{R}^d$ containing a unit line segment in every direction must have Hausdorff dimension d.

Note that $\mathcal{G}_d$ is a $2d-2$-dimensional manifold, thus X_{full} is over-determined for $d \geq 3$, and it is of interest to consider its restrictions to lower dimensional subspaces of $\mathcal{G}_d$. We consider subspaces defined by restricting the set of directions to a lower dimensional submanifold of S^{d-1}. One particular example is the restriction of X_{full} to the space of light rays (lines in $\mathbb{R}^d$ making a 45 degree angle with the plane $x_d = 0$). In [21], Wolff obtained mixed-norm estimates for this operator in all dimensions (almost sharp in $\mathbb{R}^3$ and $\mathbb{R}^4$). Also see [17] for a simplified proof of Wolff's result. Almost sharp mixed-norm estimates are also known in the cases when the set of directions is given by a curve, see [9, 10].

In this paper, we consider the X-ray transform restricted to directions lying on a 2-surface in $\mathbb{R}^{d-1}$:

$$z \to \theta(z) = (\theta_1(z), \ldots, \theta_{d-1}(z)).$$

Specifically, let f be a function on $\mathbb{R}^d$, $z \in \mathbb{R}^2$, and $y \in \mathbb{R}^{d-1}$. We define

$$T^\theta[f](z, y) = \int_0^1 f(\gamma(z, y, s)) \, ds$$

where

$$\gamma(z, y, s) = y + s(\theta(z) + e_d).$$

Here $e_1, \ldots, e_d$ is an orthonormal basis for $\mathbb{R}^d$ and we identify $\mathbb{R}^{d-1}$ with $\text{span}(e_1, \ldots, e_{d-1})$.

Letting $B \subset \mathbb{R}^2$ be a fixed ball, we have the Kakeya-order mixed-norm on the set of lines

$$\|T^\theta[f]\|_{L^q(L^r)} = \left(\int_B \left(\int_{\mathbb{R}^{d-1}} |T^\theta[f](z,y)|^r \, dy \right)^{\frac{q}{r}} dz \right)^{\frac{1}{q}}$$

and are interested in estimates

(1) $$\|T^\theta[f]\|_{L^q(L^r)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}.$$

Let s_j be the "minimum degree" of a collection of j distinct monomials in two variables: $s_1, s_2 = 1; s_3, \ldots, s_5 = 2; s_6, \ldots, s_9 = 3; s_{10}, \ldots, s_{14} = 4; \ldots$ Let $S_n = \sum_{j=1}^n s_j$. Then, for any smooth 2-surface in $\mathbb{R}^{d-1}$, (1) may only hold if the following inequalities are satisfied

(2) $$1 + \frac{d-1}{r} \geq \frac{d}{p},$$

(3) $$\frac{2}{q} + \frac{S_{d-1}}{r} \geq \frac{S_{d-1}}{p},$$

(4) $$\frac{S_{d-1}}{r} \geq \frac{S_{d-1} - 2}{p}.$$

The first necessary condition above follows by applying T^θ to the characteristic function of a δ-ball and taking δ to zero. Similarly, the second one follows by applying T^θ to the characteristic function of a parallelepiped with dimensions $1 \times \delta^{s_1} \times \delta^{s_2} \times \cdots \times \delta^{s_{d-1}}$ adapted to the cone of θ via an order s_{d-1} Taylor expansion for θ. Finally, the third one can be obtained by applying T^θ to the characteristic function of a disjoint union of $\sim \delta^{-2}$ parallelepipeds as above. Also note that (4) follows from (3) if we restrict ourself to the natural case $p \leq q$ as it was observed in [10].

We call a 2-surface "well-curved", if the corresponding restricted X-ray transform satisfies (1) for all (p, q, r) in the interior of the region determined by (2), (3), (4). We expect the following to be examples of well-curved 2-surfaces in $\mathbb{R}^{d-1}$

d	$\theta(u,v)$
4	$(u, v, u^2 + v^2)$
5	$(u, v, u^2 - v^2, uv)$
6	(u, v, u^2, uv, v^2)
7	$(u, v, u^2, uv, v^2, u^3 + v^3)$
8	$(u, v, u^2, uv, v^2, u(u^2 + v^2), v(u^2 + v^2))$
9	$(u, v, u^2, uv, v^2, u^3 + v^3, u^2v, uv^2)$

Note that the inequality (1) for critical exponents $(p, q, r) = (p_{\mathrm{cr}}(d), q_{\mathrm{cr}}(d), r_{\mathrm{cr}}(d))$ specified below

$$p_{\mathrm{cr}} = q_{\mathrm{cr}} = 1 + \frac{2(d-1)}{S_{d-1}}$$

$$r_{cr} = 1 + \frac{2d}{S_{d-1} - 2}$$

implies (1) for all possible values of (p, q, r) permitted by (2), (3), and (4) by interpolation with trivial estimates. It is important to note that

$$p_{\mathrm{cr}} \leq q_{\mathrm{cr}} \leq r_{cr}.$$

To study these operators, we use the iterated T^*T method from [5], also see [6, 18, 7, 8, 13, 15] for some related work. In [9] and [10], this method was extended to the mixed norm setting. Using a variation of the method from [9], we will prove (1) with (p, q, r) arbitrarily close to $(p_{\mathrm{cr}}(d), q_{\mathrm{cr}}(d), r_{\mathrm{cr}}(d))$ for some surfaces θ (including the well-curved example surfaces above) when $d = 4, 5, 7, 8, 9$. The case $d = 4$ was already known to hold, as shown by Wolff [21]. When $d = 6$, the method in [9] breaks down since the Jacobian is identically zero; However, by using the inflation argument from [6, 7], we are still able to obtain the almost sharp $L^p \to L^q$ estimates.

The main difference between the method here and the method in [9] is the "localization" argument. In [9], the set of directions is one-dimensional and the mixed-norm estimates are obtained by reduction to the case where the directions are localized to and well distributed within a small interval. Higher dimensional localizations were used, within the context of $L^p \to L^q$ estimates, in [7]. There, the localization was performed with respect to parallelepipeds whose axes had varying direction and varying length. Here, in order to obtain mixed-norm estimates, it seems to be necessary to fix (according to the surface θ) the directions of the axes (see Section 2). This method also gives sharp estimates for certain surfaces θ of dimension greater than 2 which are not "well-curved". However, it appears that a general result involving the mixed norm estimates is currently out of reach due to the complicated nature of the Jacobians.

In Appendix A, we present an extension of the method in [5] to the multilinear setting.

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2. BILINEAR REDUCTION

Since the operator T is local and translation-invariant in a suitable sense, and $p \leq q \leq r$, we can assume that f is supported in $B(0, 1) \subset \mathbb{R}^d$. Also, since we are not considering endpoints, it suffices to establish the restricted weak-type inequality

$$(5) \qquad \langle \chi_F, T[\chi_E] \rangle \lesssim |E|^{\frac{1}{p}} \|\chi_F\|_{L^{q'}(L^{r'})}$$

where q' and r' are dual to the exponents q and r, and $E \subset B(0, 1) \subset \mathbb{R}^d$, $F \subset B \times \mathbb{R}^{d-1}$, and B is a fixed ball in $\mathbb{R}^2$. Note that (5) is equivalent to (for

$1 < p < \infty)$

$$|\{x : T^*[\chi_F] \geq \beta\}| \lesssim \beta^{-p'} \|\chi_F\|_{L^{q'}(L^{r'})}^{p'}.$$

Therefore, in (5), we can assume that for some $0 < \tilde{\beta} \lesssim 1$, and for each $x \in E$,

(6) $$\tilde{\beta} \leq T^*[\chi_F] \leq 2\tilde{\beta}.$$

One may calculate that for $x \in \mathbb{R}^d$ and functions g on $\mathbb{R}^2 \times \mathbb{R}^{d-1}$

$$T^*[g](x) = \int_B g(\gamma^*(x, z)) \, dz$$

where

$$\gamma^*(x, z) = (z, \operatorname{proj}_{\mathbb{R}^{d-1}}(x) - \langle x, e_d \rangle \theta(z)).$$

Note that $T^*[g](x)$ is an averaging of g over the set of lines passing thru x.

2.1. Localization.
The following lemma is a variant of certain lemmas found, for example, in [21], [18], and [9].

For $R_1, R_2 > 0$, $y \in \mathbb{R}^2$, and linearly independent $w_1, w_2 \in \mathbb{R}^2$, we consider the parallelogram

$$P_{w_1, w_2}(y, R_1, R_2) = \{y + s_1 w_1 + s_2 w_2 : (s_1, s_2) \in [-R_1, R_1] \times [-R_2, R_2]\}.$$

Lemma 2.1. *Let $0 < \epsilon < 1$, and let $B \subset \mathbb{R}^2$ be a fixed ball. Then, for every $G \subset B$ with $|G| > 0$, there exist R_1, R_2 with $|G|^{1+\epsilon} \lesssim R_1, R_2 \lesssim 1$ and $y \in \mathbb{R}^2$ such that*

(7) $$|G \cap P_{w_1, w_2}(y, R_1, R_2)| \gtrsim |G|^{1+\epsilon}$$

and such that for every y' and $R_1' R_2' < R_1 R_2$

(8) $$|G \cap P_{w_1, w_2}(y', R_1', R_2')| < \left(\frac{R_1' R_2'}{R_1 R_2}\right)^{\frac{\epsilon}{2}} |G \cap P_{w_1, w_2}(y, R_1, R_2)|.$$

The implicit constants above may depend on B, ϵ, w_1, w_2.

Proof. Choose $\overline{R}_1, \overline{R}_2 \lesssim 1$ and $\overline{y}$ so that $B \subset P_{w_1, w_2}(\overline{y}, \overline{R}_1, \overline{R}_2)$, and let $\epsilon' = \frac{\epsilon}{1+\epsilon}$. Then

$$|G \cap P_{w_1, w_2}(\overline{y}, \overline{R}_1, \overline{R}_2)| \geq \left(\frac{\overline{R}_1 \overline{R}_2}{\overline{R}_1 \overline{R}_2}\right)^{\epsilon'} |G|.$$

Let R_1, R_2 be chosen with $R_1 R_2$ minimal so that there exists a y with

$$|G \cap P_{w_1, w_2}(y, R_1, R_2)| \geq \left(\frac{R_1 R_2}{\overline{R}_1 \overline{R}_2}\right)^{\epsilon'} |G|.$$

Clearly

(9) $$R_1 R_2 \gtrsim |G|^{\frac{1}{1-\epsilon'}} = |G|^{1+\epsilon}$$

By the minimality of $R_1 R_2$, and the fact that $G \subset B$, we also have $R_1, R_2 \lesssim 1$ and so $R_1, R_2 \gtrsim |G|^{1+\epsilon}$. Again from (9),

$$|G \cap P_{w_1, w_2}(y, R_1, R_2)| \gtrsim \left(\frac{|G|^{\frac{1}{1-\epsilon'}}}{\overline{R}_1 \overline{R}_2}\right)^{\epsilon'} |G| \gtrsim |G|^{1+\epsilon}.$$

The minimality of R_1, R_2 then guarantees that for every y' and $R_1' R_2' < R_1 R_2$, we have

$$|G \cap P_{w_1,w_2}(y', R_1', R_2')| < \left(\frac{R_1' R_2'}{\overline{R_1} \overline{R_2}} \right)^{\epsilon'} |G|$$

$$\leq \left(\frac{R_1' R_2'}{R_1 R_2} \right)^{\epsilon'} |G \cap P_{w_1,w_2}(y, R_1, R_2)|$$

$$\leq \left(\frac{R_1' R_2'}{R_1 R_2} \right)^{\frac{\epsilon}{2}} |G \cap P_{w_1,w_2}(y, R_1, R_2)|.$$

$\square$

2.2. **Decomposition.** We now decompose E with respect to the parallelograms obtained from Lemma 2.1. Fix w_1, w_2 depending on d (and in particular depending on the surface specified for d).

For each $x \in E$, let

$$G_x = \{z : \gamma^*(x, z) \in F\}.$$

We have G_x contained in the fixed ball B and $|G_x| = T^*[\chi_F](x) \approx \tilde{\beta}$. Applying Lemma 2.1 with $G = G_x$ and $\epsilon > 0$ (which will be determined later), we obtain $R_{1,x}, R_{2,x}$ and y_x satisfying (7) and (8). Note that for each $R_{i,x}$

$$\tilde{\beta}^{1+\epsilon} \lesssim R_{i,x} \lesssim 1$$

and so, by absorbing a possible factor of $\approx |\log(\tilde{\beta})|^2$, it suffices to show that

$$(10) \qquad \langle T^*[\chi_F], \chi_{E'} \rangle \lesssim \tilde{\beta}^\epsilon |E'|^{\frac{1}{p}} \|\chi_F\|_{L^{q'}(L^{r'})}$$

where $R_i \lesssim R_{i,x} \leq R_i$ on E'.

Cover $\mathbb{R}^2$ by parallelograms $P_{w_1,w_2}(y_j, R_1, R_2)$ which have measure-zero overlap. For each j, let

$$F_j = F \cap \left(P_{w_1,w_2}(y_j, 2R_1, 2R_2) \times \mathbb{R}^{d-1} \right)$$

and

$$E_j = \{x \in E' : y_x \in P_{w_1,w_2}(y_j, R_1, R_2)\}.$$

Then

$$(11) \qquad \langle T^*[\chi_F], \chi_{E'} \rangle \leq \sum_j \langle T^*[\chi_F], \chi_{E_j} \rangle \lesssim \tilde{\beta}^{-\epsilon} \sum_j \langle T^*[\chi_{F_j}], \chi_{E_j} \rangle$$

where the second inequality follows from (7) and the fact that $P_{w_1,w_2}(y_x, R_1, R_2) \subset P_{w_1,w_2}(y_j, 2R_1, 2R_2)$ if $y_x \in P_{w_1,w_2}(y_j, R_1, R_2)$. In Sections 3 and 4, we prove the estimate

$$(12) \qquad \langle T^*[\chi_{F_j}], \chi_{E_j} \rangle \lesssim \tilde{\beta}^{2\epsilon} |E_j|^{\frac{1}{p}} \|\chi_{F_j}\|_{L^{q'}(L^{r'})}.$$

It thus follows from Hölder's inequality that

$$(13) \qquad \sum_j \langle T^*[\chi_{F_j}], \chi_{E_j} \rangle \lesssim \tilde{\beta}^{2\epsilon} \sum_j |E_j|^{\frac{1}{p}} \|\chi_{F_j}\|_{L^{q'}(L^{r'})}$$

$$\leq \tilde{\beta}^{2\epsilon} \left(\sum_j |E_j| \right)^{\frac{1}{p}} \left(\sum_j \|\chi_{F_j}\|_{L^{q'}(L^{r'})}^{p'} \right)^{\frac{1}{p'}}.$$

We have $p' \geq q'$ and so

$$(14) \quad \Big(\sum_j |E_j|\Big)^{\frac{1}{p}} \Big(\sum_j \|\chi_{F_j}\|_{L^{q'}(L^{r'})}^{p'}\Big)^{\frac{1}{p'}} \lesssim \Big(\sum_j |E_j|\Big)^{\frac{1}{p}} \Big(\sum_j \|\chi_{F_j}\|_{L^{q'}(L^{r'})}^{q'}\Big)^{\frac{1}{q'}}$$

$$\lesssim |E'|^{\frac{1}{p}} \|\chi_F\|_{L^{q'}(L^{r'})}$$

where the last inequality follows from the finite overlap of the $P_{w_1,w_2}(y_j, 2R_1, 2R_2)$. Combining (11), (13), and (14), we obtain (10).

3. Main estimate

We now prove

$$\langle T^*[\chi_F], \chi_E \rangle \lesssim \tilde{\beta}^{2\epsilon} |E|^{\frac{1}{p}} \|\chi_F\|_{L^{q'}(L^{r'})},$$

for (p, q, r) close to $(p_{\mathrm{cr}}(d), q_{\mathrm{cr}}(d), r_{\mathrm{cr}}(d))$ (where $\epsilon > 0$ depends on $r_{\mathrm{cr}}(d) - r$), under the assumptions:

I) For some $y \in \mathbb{R}^2$ and $\tilde{\beta}^{1+\epsilon} \lesssim R_1, R_2 \lesssim 1$

$$(15) \qquad\qquad F \subset P_{w_1,w_2}(y, 2R_1, 2R_2) \times \mathbb{R}^{d-1}.$$

II) For each $x \in E$,

$$(16) \qquad\qquad \tilde{\beta}^{1+\epsilon} \lesssim T^*[\chi_F](x) \lesssim \tilde{\beta}.$$

III) For each $x \in E$, $y' \in \mathbb{R}^2$, and $R_1' R_2' < R_1 R_2$,

$$(17) \qquad T^*\Big[\chi_{F \cap \big(P_{w_1,w_2}(y', R_1', R_2') \times \mathbb{R}^{d-1}\big)}\Big](x) \lesssim \Big(\frac{R_1' R_2'}{R_1 R_2}\Big)^{\frac{\epsilon}{2}} T^*[\chi_F](x).$$

By Section 2, this suffices to prove (1).

Absorbing a possible factor of $\approx |\log(\tilde{\beta})|$, we assume without loss of generality that $T^*[\chi_F] \approx \beta'$ on E where $\tilde{\beta}^{1+\epsilon} \lesssim \beta' \lesssim \tilde{\beta}$.

The quantities

$$\alpha = \frac{\langle T^*[\chi_F], \chi_E \rangle}{|F|}$$

and

$$\beta = \frac{\langle T^*[\chi_F], \chi_E \rangle}{|E|}$$

will appear throughout this section. Of course $\beta \approx \beta'$.

3.1. Iterated maps and parameter-space towers.
Let n be the integer satisfying $d + 1 \leq 3n \leq d + 3$. Fix a line (z_0, y_0) to be specified below, and define the maps

$$\Phi_1(t_1) = \gamma(z_0, y_0, t_1)$$

and

$$\Phi_1^*(t_1, z_1) = \gamma^*(\Phi_1(t_1), z_1).$$

For $i = 2, \ldots, n$ define the iterated maps

$$\Phi_i(t_1, z_1, \ldots, t_{i-1}, z_{i-1}, t_i) = \gamma(\Phi_{i-1}^*(t_1, z_1, \ldots, t_{i-1}, z_{i-1}), t_i)$$

and

$$\Phi_i^*(t_1, z_1, \ldots, t_i, z_i) = \gamma^*(\Phi_i(t_1, z_1, \ldots, t_{i-1}, z_{i-1}, t_i), z_i).$$

For any $\Omega \subset \left(\mathbb{R}^1 \times \mathbb{R}^2\right)^n$ and $1 \leq i \leq n$, let

$$\Omega_i^* = \{(t_1, z_1, \ldots, t_i, z_i) : (t_1, z_1, \ldots, t_n, z_n) \in \Omega\},$$
$$\Omega_i = \{(t_1, z_1, \ldots, t_{i-1}, z_{i-1}, t_i) : (t_1, z_1, \ldots, t_n, z_n) \in \Omega\}.$$

Definition 3.1. *Let $\overline{\alpha}, \overline{\beta} > 0$, and $\Omega \subset \left(\mathbb{R}^1 \times \mathbb{R}^2\right)^n$. We say that Ω is an $(\overline{\alpha}, \overline{\beta})$ tower if there exists a $(z_0, y_0) \in F$ so that the following conditions hold.*

(18) $$|\Omega_1| \geq 2^{-4n}\overline{\alpha}.$$

For $1 < i \leq n$

(19) $$|s : (\omega^*, s) \in \Omega_i| \geq 2^{-4n}\overline{\alpha} \text{ for every } \omega^* \in \Omega_{i-1}^*.$$

For $1 \leq i \leq n$

(20) $$\Phi_i(\omega) \in E \text{ for every } \omega \in \Omega_i$$

(21) $$|z : (\omega, z) \in \Omega_i^*| \geq 2^{-4n}\overline{\beta} \text{ for every } \omega \in \Omega_i$$

(22) $$\Phi_i^*(\omega^*) \in F \text{ for every } \omega^* \in \Omega_i^*.$$

The following is essentially Lemma 1 of [5].

Lemma 3.2. *There exists an (α, β) tower.*

3.2. Change of variables. Let Ω be an (α', β') tower where $\alpha' \gtrsim \alpha$ and $\beta' \gtrsim \beta$.

Fix $t_1 \in \Omega_1$. If $3n = d + 3$ let

$$\Pi = \{(z_1, t_2, z_2, \ldots, t_{n-1}, z_{n-1}, t_n) : (t_1, z_1, \ldots, t_n, z_n) \in \Omega\},$$

if $3n = d + 2$ let

$$\Pi = \{(z_1, t_2, z_2, \ldots, t_n, z_n) : (t_1, z_1, \ldots, t_n, z_n) \in \Omega\},$$

and if $3n = d + 1$ let

$$\Pi = \Omega.$$

Set $\Phi = \Phi_n(t_1, \cdot)$ if $3n = d + 3$, $\Phi = \Phi_n^*(t_1, \cdot)$ if $3n = d + 2$, and $\Phi = \Phi_n^*$ if $3n = d + 1$.

Then

$$\Phi : \Pi \to E$$

if $3n = d + 3$ and

$$\Phi : \Pi \to F$$

if $3n = d + 2, d + 1$.

From Bezout's theorem, these mappings are generically finite-to-one. Thus,

(23) $$|E| \gtrsim \int_\Pi J(\omega) \, d\omega$$

if $3n = d + 3$ where $J = |\det(\partial\Phi/\partial u_1, v_1, \ldots, t_n)|$, and

(24) $$|F| \gtrsim \int_\Pi J(\omega) \, d\omega$$

if $3n = d + 1, d + 2$ where

$$J = |\det(\partial\Phi/\partial u_1, v_1, \ldots, t_n, u_n, v_n)| \text{ if } 3n = d + 2$$

and
$$J = |\det(\partial\Phi/\partial t_1, u_1, v_1, \ldots, t_n, u_n, v_n)| \text{ if } 3n = d+1.$$
Above, we write $z_i = (u_i, v_i)$.

In Section 4, we show that Ω may be chosen so that

(25) $$|J| \gtrsim \alpha^{k_d}\beta^{l_d}R_1R_2$$

on Ω where k_d and l_d are given by

$$k_d = 2(n-1), \quad l_d = \frac{S_{d-1}}{2} - n.$$

Since Ω is an (α', β') tower with $\alpha' \gtrsim \alpha$, and $\beta' \gtrsim \beta$, we have

$$\begin{aligned}
|\Pi| &\gtrsim \alpha^{n-1}\beta^{n-1}, &&\text{if } 3n = d+3,\\
|\Pi| &\gtrsim \alpha^{n-1}\beta^{n}, &&\text{if } 3n = d+2,\\
|\Pi| &\gtrsim \alpha^{n}\beta^{n}, &&\text{if } 3n = d+1.
\end{aligned}$$

It thus follows from (23), (24), (25), and the definitions of α and β that

(26) $$R_1R_2\langle\chi_F, T[\chi_E]\rangle^{\frac{S_{d-1}}{2}+d-1} \lesssim |E|^{\frac{S_{d-1}}{2}}|F|^d.$$

From (15) and Hölder's inequality, it follows that

(27) $$\left(\frac{|F|}{(R_1R_2)^{1-\frac{r'}{q'}}}\right)^{\frac{1}{r'}} \lesssim \|\chi_F\|_{L^{q'}(L^{r'})}.$$

Thus, combining (26) and (27), we obtain

$$\langle\chi_F, T[\chi_E]\rangle \lesssim |E|^{\frac{1}{p_{\mathrm{cr}}}}\|\chi_F\|_{L^{q'_{\mathrm{cr}}}(L^{r'_{\mathrm{cr}}})}.$$

This implies that

$$\langle\chi_F, T[\chi_E]\rangle \lesssim \beta^{2\epsilon}|E|^{\frac{1}{p}}\|\chi_F\|_{L^{q'}(L^{r'})}$$

where (p, q, r) are given by an arbitrarily small interpolation of $(p_{\mathrm{cr}}, q_{\mathrm{cr}}, r_{\mathrm{cr}})$ with $(1, 1, 1)$.

4. Lower bounds for the Jacobians

Let Ω^0 be the (α, β) tower guaranteed by Lemma 3.2. For any fixed $t_1, \ldots, t_i$,

$$|\{t : \min_{j=1,\ldots,i}|t - t_j| \ll \alpha\}| \ll \alpha.$$

Thus, by induction, one may find an (α', β) tower $\Omega^1 \subset \Omega^0$, with $\alpha' \gtrsim \alpha$ so that $|t_i - t_j| \gtrsim \alpha$ for $(t_1, z_1, \ldots, t_n, z_n) \in \Omega^1$ and $1 \le i < j \le n$.

Additional refinements of Ω^1 needed to bound the Jacobian will have to be tailored to the individual 2-surface in question. However, we will use the following lemma repeatedly.

Lemma 4.1. *Let $0 < C < 1$, $\overline{\alpha} > 0$, and let Ω be an $(\overline{\alpha}, C\beta)$ tower. Let $1 \le i \le n$, and let $\{P_{w_1,w_2}(y_\omega, R'_1, R'_2)\}_{\omega\in\Omega_i}$ be a family of parallelograms with $R'_1R'_2 \ll R_1R_2(C)^{\frac{2}{\epsilon}}$. Then*

$$\Omega' = \{(\omega, z_i, \ldots, t_n, z_n) \in \Omega : z_i \notin P_{w_1,w_2}(y_\omega, R'_1, R'_2)\}$$

is an $(\overline{\alpha}, \frac{1}{2}C\beta)$ tower.

Proof. It suffices to check that for each $\omega \in \Omega_1$,

$$|\{z_i \notin P_{w_1,w_2}(y_\omega, R_1', R_2') : (\omega, z_i, \ldots, t_n, z_n) \in \Omega\}| \geq 2^{-4n}\frac{1}{2}C\beta$$

which follows from

$$(28) \qquad |\{z_i \in P_{w_1,w_2}(y_\omega, R_1', R_2') : (\omega, z_i, \ldots, t_n, z_n) \in \Omega\}| < 2^{-4n}\frac{1}{2}C\beta.$$

To see (28), note that, from (22), we have

$$\{z_i \in P_{w_1,w_2}(y_\omega, R_1', R_2') : (\omega, z_i, \ldots, t_n, z_n) \in \Omega\} \subset \{z_i : \gamma^*(\Phi_i(\omega), z_i) \in F\}.$$

But, from (20), we have $\Phi_i(\omega) \in E$ and so from (17)

$$|\{z_i : \gamma^*(\Phi_i(\omega), z_i) \in F\}| \lesssim \left(\frac{R_1'R_2'}{R_1R_2}\right)^{\frac{\epsilon}{2}}\beta'$$

Since $\beta' \lesssim \beta$, we thus have (28) from our choice of $R_1'R_2'$. $\qquad\square$

4.1. **The case** $3n = d+1$**.** One may calculate that for $\omega = (t_1, z_1, \ldots, t_n, z_n)$ and $z_i = (u_i, v_i)$

$$\Phi(\omega) = \left(z_n, \ y_0 + \sum_{i=1}^{n} t_i(\theta(z_{i-1}) - \theta(z_i))\right).$$

Thus,

$$J = |\det(\partial\Phi/\partial t_1, u_1, v_1, \ldots, t_n, u_n, v_n)|$$
$$= J_t \cdot J_z$$

where

$$(29) \qquad\qquad J_t = \prod_{i=1}^{n-1}(t_{i+1} - t_i)^2, \quad \text{and}$$

$$(30) \quad J_z = |\det(\theta_u(z_1), \theta_v(z_1), \ldots, \theta_u(z_{n-1}), \theta_v(z_{n-1}),$$
$$\theta(z_0) - \theta(z_1), \ldots, \theta(z_{n-1}) - \theta(z_n))|.$$

Above, we denote $\theta_u = \frac{\partial\theta}{\partial u}$ and $\theta_v = \frac{\partial\theta}{\partial v}$.

On Ω^1, we have $J_t \gtrsim \alpha^{2(n-1)}$. Thus, it remains to find an (α', β') tower $\Omega \subset \Omega^1$ with $J_z \gtrsim R_1R_2\beta^{l_d}$ on Ω and $\beta' \gtrsim \beta$.

4.1.1. *d=5.* We now work under the assumption that $d = 5$ and that θ is of the form

$$\theta(u, v) = (u, v, \overline{\theta}(u, v)).$$

We may then simplify (30) to

$$J_z = |\det(\overline{\theta}(z_0) - \overline{\theta}(z_1) - (u_0 - u_1)\overline{\theta}_u(z_1) - (v_0 - v_1)\overline{\theta}_v(z_1),$$
$$\overline{\theta}(z_2) - \overline{\theta}(z_1) - (u_2 - u_1)\overline{\theta}_u(z_1) - (v_2 - v_1)\overline{\theta}_v(z_1))|.$$

Assuming that the entries in $\overline{\theta}$ are polynomials of degree 2 or less, this simplifies to

$$J_z = \left|\det\left(\frac{\overline{u}_0^2\overline{\theta}_{uu}}{2} + \frac{\overline{v}_0^2\overline{\theta}_{vv}}{2} + \overline{u}_0\overline{v}_0\overline{\theta}_{uv}, \ \frac{\overline{u}_2^2\overline{\theta}_{uu}}{2} + \frac{\overline{v}_2^2\overline{\theta}_{vv}}{2} + \overline{u}_2\overline{v}_2\overline{\theta}_{uv}\right)\right|$$

where
$$\overline{z}_0 = z_1 - z_0, \ \overline{z}_2 = z_2 - z_1.$$
After some algebra, we obtain

$$(31) \qquad J_z = \left| (\overline{u}_0, \overline{v}_0) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \overline{u}_2 \\ \overline{v}_2 \end{pmatrix} \right| \cdot \left| (\overline{u}_0, \overline{v}_0) \begin{pmatrix} A & \frac{B}{2} \\ \frac{B}{2} & C \end{pmatrix} \begin{pmatrix} \overline{u}_2 \\ \overline{v}_2 \end{pmatrix} \right|$$

where
$$A = \frac{1}{2} \det(\overline{\theta}_{uu}, \overline{\theta}_{uv}), \ B = \frac{1}{2} \det(\overline{\theta}_{uu}, \overline{\theta}_{vv}), \ C = \frac{1}{2} \det(\overline{\theta}_{uv}, \overline{\theta}_{vv}).$$

We need the following lemma to further refine the parameter space tower.

Lemma 4.2. *Let $A, B, C \in \mathbb{R}$ satisfy $B^2 - 4AC \neq 0$, let w_1, w_2 be chosen, as specified below, according to A, B, C, let $0 \leq i < j \leq n$ and let Ω be an α', β' tower with $\beta' \gtrsim \beta$. Then, there is an (α', β'') tower $\Omega' \subset \Omega$ with $\beta'' \gtrsim \beta'$ so that*
$$|Q_{i,j}| \gtrsim R_1 R_2$$
on Ω' where
$$Q_{i,j}(t_1, \ldots, z_n) := A(u_j - u_i)^2 + B(u_j - u_i)(v_j - v_i) + C(v_j - v_i)^2.$$

Proof. Let $z = z_j - z_i$. Assume first that $B^2 - 4AC > 0$.
  If $A \neq 0$, we have
$$Q_{i,j} = A(u - D_+ v)(u - D_- v)$$
where
$$D_+ = \frac{-B + \sqrt{B^2 - 4AC}}{2A}, \ \text{and} \ D_- = \frac{-B - \sqrt{B^2 - 4AC}}{2A}.$$
Then taking $w_1 = (D_+, 1)$ and $w_2 = (D_-, 1)$ we may apply Lemma 4.1 twice to find Ω' with $|u - D_+ v| \gtrsim R_2$ and $|u - D_- v| \gtrsim R_1$ on Ω'.
  If $A = 0$, we have $B \neq 0$ and
$$Q_{i,j} = v(Bu + Cv).$$
Then taking $w_1 = (1, 0), w_2 = (-C, B)$, we may apply Lemma 4.1 twice to find Ω' with $|v| \gtrsim R_2$ and $|Bu + Cv| \gtrsim R_1$ on Ω'.
  If $B^2 - 4AC < 0$
$$|Q_{i,j}| = |v|^2 \left| A\left(\frac{u}{v}\right)^2 + B\frac{u}{v} + C \right| \geq |v|^2 \left| \frac{B^2 - 4AC}{4A} \right|.$$
Similarly $|Q_{i,j}| \gtrsim |u|^2$, and thus $|Q_{i,j}| \gtrsim |z|$. Then taking any w_1, w_2 we may apply Lemma 4.1 twice to find Ω' with $|z| \gtrsim \max(R_1, R_2)^2$ on Ω'. $\square$

We now assume the non-degeneracy condition (which is equivalent to the condition from [4], as was pointed out to the second author by D. Oberlin) $B^2 - 4AC \neq 0$. From Lemma 4.2, we may find an (α', β') tower $\Omega^2 \subset \Omega^1$ with $\beta' \gtrsim \beta$ so that
$$Q(\overline{z}_0) \gtrsim R_1 R_2$$
for $(z_0, z_1) \in \Omega^2$ where
$$Q(\overline{z}_0) = \left| (\overline{u}_0, \overline{v}_0) \begin{pmatrix} A & \frac{B}{2} \\ \frac{B}{2} & C \end{pmatrix} \begin{pmatrix} \overline{u}_0 \\ \overline{v}_0 \end{pmatrix} \right|.$$

Choose $i \neq j \in \{1, 2\}$ so that $R_i \geq R_j$. Refining further, find $\Omega^3 \subset \Omega^2$ so that $|\overline{z}_0| \approx |\overline{z}_2| \approx R_i$ on Ω_3.

Let
$$\overline{z}_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A & \frac{B}{2} \\ \frac{B}{2} & C \end{pmatrix} \begin{pmatrix} \overline{u}_0 \\ \overline{v}_0 \end{pmatrix}.$$

Then, since $|\overline{z}_0| \approx |\overline{z}_1| \approx R_i$ and
$$|\det(\overline{z}_0, \overline{z}_1)| = Q(\overline{z}_0) \gtrsim R_1 R_2,$$

we have $|\sin(\rho)| \gtrsim \frac{R_j}{R_i}$ where ρ is the angle between $\overline{z}_0$ and $\overline{z}_1$.

Let ρ_{02} be the angle between $\overline{z}_2$ and $\overline{z}_0$ and let ρ_{12} be the angle between $\overline{z}_2$ and $\overline{z}_1$. Then

$$J_z = |\overline{z}_0||\overline{z}_1||\overline{z}_2|^2 |\sin(\rho_{02})||\sin(\rho_{12})|$$
$$\approx R_i^4 |\sin(\rho_{02})||\sin(\rho_{12})|.$$

For each $\overline{z}_0$, choose $D_{\overline{z}_0}$ so that $\min(|\sin(\rho_{02})|, |\sin(\rho_{12})|) \leq D_{\overline{z}_0}$ for exactly half of the $\overline{z}_2$. Note that $D_{\overline{z}_0} \gtrsim \frac{\beta}{R_i^2}$. If $D_{\overline{z}_0} \gtrsim \frac{R_j}{R_i}$ then, for the other half of the $\overline{z}_2$, we have

$$|\sin(\rho_{02})||\sin(\rho_{12})| \gtrsim \frac{R_j^2}{R_i^2}$$

and so $J_z \gtrsim (R_1 R_2)^2 \geq R_1 R_2 \beta$ for half of the $\overline{z}_2$.

Since $|\sin(\rho)| \gtrsim \frac{R_j}{R_i}$, we have $|\sin(\rho_{12})| \gtrsim \frac{R_j}{R_i}$ if $|\sin(\rho_{02})| \ll \frac{R_j}{R_i}$ and we have $|\sin(\rho_{02})| \gtrsim \frac{R_j}{R_i}$ if $|\sin(\rho_{12})| \ll \frac{R_j}{R_i}$. Thus, if $D_{\overline{z}_0} \ll \frac{R_j}{R_i}$, we have

$$|\sin(\rho_{02})||\sin(\rho_{12})| \gtrsim \frac{R_j}{R_i} D_{\overline{z}_0} \gtrsim \frac{R_j}{R_i} \frac{\beta}{R_i^2}$$

and thus
$$J_z \gtrsim \beta R_1 R_2$$

for half of the $\overline{z}_2$.

In either case, we may find an (α', β'') tower $\Omega^4 \subset \Omega^3$ with $\beta'' \gtrsim \beta$ so that $J_z \gtrsim \beta R_1 R_2$ on Ω^4.

4.1.2. d=8. In this case, we assume that θ is of the form
$$\theta(u, v) = (u, v, u^2, uv, v^2, u(u^2 + v^2), v(u^2 + v^2)).$$

We simplify[1] the Jacobian (30) to

$$J_z = \big| \det(\overline{z}_2, \overline{z}_3) \det(\overline{z}_1, \overline{z}_2) |\overline{z}_2|^2 \big|$$
$$\times \big| \det(\overline{z}_3, \overline{z}_1 + \overline{z}_2) |\overline{z}_2 + \overline{z}_3|^2 + \det(\overline{z}_2, \overline{z}_3) |\overline{z}_1 + \overline{z}_2 + \overline{z}_3|^2 + \det(\overline{z}_1, \overline{z}_2 + \overline{z}_3) |\overline{z}_3|^2 \big|,$$

where $\overline{z}_j := z_{j-1} - z_j$ for $j = 1, 2, 3$. Let ρ_{ij} be the clockwise angle $\overline{z}_i$ to $\overline{z}_j$. After some algebra, we can rewrite

$$J_z = |\overline{z}_1|^2 |\overline{z}_2|^5 |\overline{z}_3|^2 |\sin(\rho_{23}) \sin(\rho_{12})|$$
$$\times \Big| \sin(\rho_{12}) |\overline{z}_3| + \sin(\rho_{23}) \frac{|\overline{z}_1 + \overline{z}_2 + \overline{z}_3|^2 - |\overline{z}_2 + \overline{z}_3|^2}{|\overline{z}_1|} + \sin(\rho_{31}) \frac{|\overline{z}_2 + \overline{z}_3|^2 - |\overline{z}_3|^2}{|\overline{z}_2|} \Big|.$$

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[1]We used Maple to simplify the Jacobians

Using that for any i, j, k, $\rho_{ij} + \rho_{jk} = \rho_{ik}$, $\langle \overline{z}_i, \overline{z}_j \rangle = |z_i||z_j| \cos(\rho_{ij})$, and trigonometric identities we write

$$J_z = |\overline{z}_1|^2 |\overline{z}_2|^5 |\overline{z}_3|^2 \big| \sin(\rho_{23}) \sin(\rho_{12}) \big| \big| |\overline{z}_1| \sin(\rho_{23}) + |\overline{z}_2| \sin(\rho_{23} + \rho_{21}) + |\overline{z}_3| \sin(\rho_{21}) \big|.$$

To obtain the required lower bound for J_z, it suffices to use a localization to squares. Namely, in Lemma 2.1 and Lemma 4.1, we let w_1, w_2 be the coordinate axis directions and require that $R_1 = R_2 = R$. Note that using Lemma 4.1 as in Lemma 4.2, we can guarantee that we have an (α', β') tower $\Omega^2 \subset \Omega^1$ with $\beta' \gtrsim \beta$ so that

$$|\overline{z}_j| \gtrsim R, \quad \text{and} \quad |\sin(\rho_{ij})| \gtrsim \beta/R^2$$

on Ω^2. This implies that

$$J_z \gtrsim R^5 \beta^2 \big| |\overline{z}_1| \sin(\rho_{23}) + |\overline{z}_2| \sin(\rho_{23} + \rho_{21}) + |\overline{z}_3| \sin(\rho_{21}) \big|.$$

To estimate the remaining term we have to refine Ω^2 once more. Note that for each fixed $\overline{z}_1, \overline{z}_2$ and a fixed argument for $\overline{z}_3$, we have

$$\big| |\overline{z}_1| \sin(\rho_{23}) + |\overline{z}_2| \sin(\rho_{23} + \rho_{21}) + |\overline{z}_3| \sin(\rho_{21}) \big| \gtrsim (\beta/R)|\sin(\rho_{21})| \gtrsim \beta^2/R^3$$

for each value of $|\overline{z}_3|$ except for $|\overline{z}_3|$ in an interval of length $\ll (\beta/R)$. Therefore by removing a set of measure $\ll \beta$ for $\overline{z}_3$, we can find an (α', β'') tower $\Omega \subset \Omega^2$ with $\beta'' \gtrsim \beta$ on which

$$J_z \gtrsim R^2 \beta^4 = R_1 R_2 \beta^4.$$

4.2. The case $3n = d + 2$. One may calculate that for $\omega = (z_1, \ldots, t_n, z_n)$ and $z_i = (u_i, v_i)$

$$\Phi(\omega) = \Big(z_n, \ y_0 + \sum_{i=1}^{n} t_i(\theta(z_{i-1}) - \theta(z_i)) \Big).$$

Thus,

$$J = |\det(\partial \Phi / \partial u_1, v_1, \ldots, t_n, u_n, v_n)| = J_t \cdot J_z$$

where

$$J_t = \prod_{i=1}^{n-1} (t_{i+1} - t_i)^2, \quad \text{and}$$

$$(32) \quad J_z = |\det(\theta_u(z_1), \theta_v(z_1), \ldots, \theta_u(z_{n-1}), \theta_v(z_{n-1}),$$
$$\theta(z_1) - \theta(z_2), \ldots, \theta(z_{n-1}) - \theta(z_n))|.$$

On Ω^1, we have $J_t \gtrsim \alpha^{2(n-1)}$. Thus, it remains to find an (α', β') tower $\Omega \subset \Omega^1$ with $J_z \gtrsim R_1 R_2 \beta^{l_d}$ on Π and $\beta' \gtrsim \beta$.

4.2.1. $d=7$. In this case, we assume that θ is of the form

$$\theta(u, v) = (u, v, u^2, uv, v^2, P(u, v)),$$

where $P(u, v) = au^3 + bu^2v + cuv^2 + dv^3$. We simplify the Jacobian (32) to

$$J_z = \big| \det(\overline{z}_1, \overline{z}_2) \big|^2 |P(\overline{z}_1)|,$$

where $\overline{z}_j := z_{j+1} - z_j$. At this point we assume that $|P(\overline{z}_1)| \gtrsim |L(\overline{z}_1)||\overline{z}_1|^2$, where $L(\overline{z}_1) = \langle \overline{z}_1, (a, b) \rangle$ for some vector (a, b). For example, with $\theta(u, v) = (u, v, u^2, uv, v^2, u^3 + v^3)$ we may take $(a, b) = (1, 1)$.

We apply Lemma 3.2 and Lemma 4.1 with $w_1 = (a, b)$ and $w_2 \perp w_1$, and refine further as in the previous cases to obtain a (α', β') tower Ω such that $\beta' \gtrsim \beta$, and

$$|\overline{z}_1| \gtrsim \max(R_1, R_2),$$
$$|\det(\overline{z}_1, \overline{z}_2)| \gtrsim \beta,$$
$$|L(\overline{z}_1)| \gtrsim \min(R_1, R_2).$$

This implies that

$$J_z \gtrsim \beta^2 R_1 R_2 \max(R_1, R_2) \gtrsim \beta^{5/2} R_1 R_2.$$

4.3. **The case** $3n = d + 3$. One may calculate that for $\omega = (z_1, t_2, \dots, z_{n-1}, t_n)$ and $z_i = (u_i, v_i)$

$$\Phi(\omega) = y_0 + \sum_{i=1}^{n-1} t_i(\theta(z_{i-1}) - \theta(z_i)) + t_n(\theta(z_{n-1}) + e_d).$$

Thus,

$$J = |\det(\partial\Phi/\partial u_1, v_1, \dots, u_{n-1}, v_{n-1}, t_n)| = J_t \cdot J_z$$

where

$$J_t = \prod_{i=1}^{n-1} (t_{i+1} - t_i)^2, \quad \text{and}$$

$$(33) \quad J_z = |\det(\theta_u(z_1), \theta_v(z_1), \dots, \theta_u(z_{n-1}), \theta_v(z_{n-1}),$$
$$\theta(z_1) - \theta(z_2), \dots, \theta(z_{n-2}) - \theta(z_{n-1}))|.$$

On Ω^1, we have $J_t \gtrsim \alpha^{2(n-1)}$. Thus, it remains to find an (α', β') tower $\Omega \subset \Omega^1$ with $J_z \gtrsim R_1 R_2 \beta^{l_d}$ on Π and $\beta' \gtrsim \beta$.

4.3.1. *d=9.* In this case, we assume that θ is of the form

$$\theta(u, v) = (u, v, u^2, uv, v^2, u^3 + v^3, u^2 v, uv^2),$$

We simplify the Jacobian (33) to

$$J_z = |\det(\overline{z}_1, \overline{z}_2)|^4 |\overline{u}_3 \overline{u}_2 \overline{u}_1 + \overline{v}_3 \overline{v}_2 \overline{v}_1|,$$

where $\overline{z}_1 := z_2 - z_1$, $\overline{z}_2 := z_3 - z_1$, and $\overline{z}_3 := z_3 - z_2$. We apply Lemma 3.2 and Lemma 4.1 with $w_1 = (0, 1)$ and $w_2 = (1, -1)$, and refine further as in previous cases to obtain a (α', β') tower Ω^2 such that $\beta' \gtrsim \beta$, and

$$(34) \qquad\qquad |\overline{z}_j| \sim \max(R_1, R_2),$$
$$(35) \qquad\qquad |\overline{u}_j| \sim R_2,$$
$$(36) \qquad\qquad |1 + \overline{v}_1/\overline{u}_1| \geq c_1 R_1/R_2,$$
$$(37) \qquad\qquad |\rho_{12}| \gtrsim \beta/\max(R_1, R_2)^2,$$

where $\rho_{ij} \in [-\pi/2, \pi/2]$ is the angle between $\overline{z}_i$ and $\overline{z}_j$.
Case 1: $R_1 \gg R_2$. By (34) and (35), we have $|\overline{v}_j| > 2|\overline{u}_j|$, which implies that

$$|\overline{u}_3 \overline{u}_2 \overline{u}_1 + \overline{v}_3 \overline{v}_2 \overline{v}_1| \gtrsim |\overline{v}_3 \overline{v}_2 \overline{v}_1| \gtrsim R_1^3.$$

Also, by (34) and (37), we have $|\det(\overline{z}_1, \overline{z}_2)| \gtrsim \beta$. Therefore,

$$J_z \gtrsim \beta^4 R_1^3 \gtrsim \beta^{9/2} R_1 R_2.$$

Case 2: $R_2 \gtrsim R_1$, and $\beta/R_2^2 \lesssim |\rho_{12}| \leq c_2 R_1/R_2$. Here $c_2 \ll c_1$ is a fixed small constant.

Note that since $|\overline{z}_j| \sim R_2$ for each j, $|\rho_{12}| \leq c_2 R_1/R_2$, and $\overline{z}_2 = \overline{z}_1 + \overline{z}_3$, by choosing c_2 sufficiently small, we have $|\rho_{13}| \leq c_3 R_1/R_2$, where $c_3 \ll c_1$.

We have two subcases $1 + \overline{v}_1/\overline{u}_1 \leq -c_1 R_1/R_2$ and $1 + \overline{v}_1/\overline{u}_1 \geq c_1 R_1/R_2$. In the former case, since $|\rho_{13}|, |\rho_{12}| \ll c_1 R_1/R_2$, we have

$$\overline{v}_j/\overline{u}_j \leq -1 - c_4 R_1/R_2,$$

for each j. Therefore,

$$(38) \qquad J(z) \gtrsim \beta^4 R_2^3 \left| 1 + \frac{\overline{v}_1}{\overline{u}_1} \frac{\overline{v}_2}{\overline{u}_2} \frac{\overline{v}_3}{\overline{u}_3} \right| = \beta^4 R_2^3 \left(-\frac{\overline{v}_1}{\overline{u}_1} \frac{\overline{v}_2}{\overline{u}_2} \frac{\overline{v}_3}{\overline{u}_3} - 1 \right)$$

$$\geq \beta^4 R_2^3 ((1 + c_4 R_1/R_2)^3 - 1) \gtrsim \beta^4 R_2^3 R_1/R_2 \gtrsim \beta^{9/2} R_1 R_2.$$

In the latter case, we can additionally assume that $\overline{v}_1/\overline{u}_1 \leq 1/4$ (Otherwise $\overline{v}_j/\overline{u}_j \gtrsim 1$ for each j by the arguments above. This case can be handled as in case 1). As above, we now have (if c_2 is sufficiently small)

$$1/2 \geq \overline{v}_j/\overline{u}_j \geq -1 + c_4 R_1/R_2$$

for each j. As in (38), we have

$$J(z) \gtrsim \beta^4 R_2^3 \left| 1 + \frac{\overline{v}_1}{\overline{u}_1} \frac{\overline{v}_2}{\overline{u}_2} \frac{\overline{v}_3}{\overline{u}_3} \right| \geq \beta^4 R_2^3 \left(1 - \left| \frac{\overline{v}_1}{\overline{u}_1} \frac{\overline{v}_2}{\overline{u}_2} \frac{\overline{v}_3}{\overline{u}_3} \right| \right)$$

$$\geq \beta^4 R_2^3 \left(1 - \left| \frac{\overline{v}_1}{\overline{u}_1} \right| \right) \gtrsim \beta^4 R_2^3 R_1/R_2 \gtrsim \beta^{9/2} R_1 R_2.$$

Case 3: $R_2 \gtrsim R_1$, and $|\rho_{12}| \gtrsim R_1/R_2$. First note that

$$(39) \qquad\qquad\qquad \left| \det(\overline{z}_1, \overline{z}_2) \right| \gtrsim R_2^2 R_1/R_2 = R_1 R_2.$$

Now we estimate the remaining term in $J(z)$. With the previous notation $\overline{z}_1 := z_2 - z_1$, $\overline{z}_2 := z_3 - z_1$, and $\overline{z}_3 := z_3 - z_2$, we have

$$\left| \overline{u}_3 \overline{u}_2 \overline{u}_1 + \overline{v}_3 \overline{v}_2 \overline{v}_1 \right| = |u_2 - u_1| \left| u_3^2 - u_3(u_2 + u_1) + g(z_1, z_2, v_3) \right|$$

$$\sim R_2 \left| u_3^2 - u_3(u_2 + u_1) + g(z_1, z_2, v_3) \right| =: R_2 |\tilde{J}(z)|.$$

By refining Ω^2 once again, we have

$$(40) \qquad\qquad\qquad \left| 2u_3 - (u_2 + u_1) \right| \gtrsim R_2.$$

Note that $\tilde{J}$ is a quadratic polynomial in u_3 satisfying (by (40))

$$|\tilde{J}_{u_3}| \gtrsim R_2.$$

Therefore, for each fixed z_1, z_2, and v_3, by removing an interval of length $\sim \beta/R_2$ for u_3, we have

$$|\tilde{J}| \gtrsim R_2 \beta/R_2 = \beta.$$

Combining the above estimates, we obtain

$$J(z) \gtrsim (R_1 R_2)^4 R_2 \beta \gtrsim \beta^{9/2} R_1 R_2.$$

4.3.2. *d=6.* In this case the Jacobian turn out to be identically zero. To obtain $L^p \to L^q$ estimates we utilize the inflation argument from [6, 7]. We only give a sketch of the argument. First, we replace the parameter space tower Π (after the change of variables) with the following "parameter space tree":

$$\bar{\Pi} = \{(z_1, t_{12}, z_{12}, t_{13}, z_{13}, t_{22}, z_{22}, t_{23}, z_{23}) :$$
$$(t_1, z_1, t_{12}, z_{12}, t_{13}, z_{13}), (t_1, z_1, t_{22}, z_{22}, t_{23}, z_{23}) \in \Omega\}$$

which is a subset of $\mathbb{R}^{14}$ of measure $\gtrsim \alpha^4 \beta^5$. We also define the inflated map $\varphi : \bar{\Pi} \to F \times F \subset \mathbb{R}^{14}$ as

$$\varphi(z_1, t_{12}, \ldots, z_{23}) = (\Phi_3^*(t_1, z_1, t_{12}, z_{12}, t_{13}, z_{13}), \Phi_3^*(t_1, z_1, t_{22}, z_{22}, t_{23}, z_{23})).$$

For the definition of Φ_3^*, see Section 3.1. As before we need to find a lower bound for the Jacobian of φ which is valid on a subset of $\bar{\Pi}$ of comparable measure. One can write the Jacobian J as $J_t \cdot J_z$ where

$$J_t = (t_{12} - t_1)(t_{22} - t_1)(t_{13} - t_{12})^2(t_{23} - t_{22})^2,$$

$$J_z = \det(z_1 - z_{22}, z_1 - z_{12}) \det(z_1 - z_{12}, z_{13} - z_{12})^2 \det(z_1 - z_{22}, z_{23} - z_{22})^2.$$

We can obtain the following lower bounds for J_t and J_z on a subset of $\bar{\pi}$ of measure $\gtrsim \alpha^4 \beta^5$ by considerations as above:

$$|J_t| \gtrsim \alpha^6,$$

$$|J_z| \gtrsim \beta^5.$$

This implies that

$$|F| = \sqrt{|F \times F|} \gtrsim \sqrt{\alpha^4 \beta^5 \alpha^6 \beta^5} = \alpha^5 \beta^5.$$

Using the definition of α and β, we obtain all $L^p \to L^q$ estimates for $(1/p, 1/q)$ in the interior of the convex hull of the points $(1/2, 2/5), (0, 1), (1, 0), (1, 1)$, which is essentially optimal.

Appendices

A. Multi-linear estimates

In this appendix, we present an extension of the method in [5] to the multi-linear setting. We only discuss various model cases.

A.1. Restricted directions. Let θ be the moment curve $\theta(u) = (u, u^2, \ldots, u^{d-1})$. Suppose I_1 and I_2 are disjoint compact intervals, and g_j is a function on $I_j \times \mathbb{R}^{d-1}$, $j = 1, 2$.

We can consider the (adjoint) bilinear X-ray estimate

$$\|T^*[g_1]T^*[g_2]\|_{L^{\frac{p'}{2}}} \lesssim \|g_1\|_{L^{q'}} \|g_2\|_{L^{q'}}.$$

From the δ-ball counterexample, we have the usual necessary condition

(41) $$\frac{d-1}{q'} \leq \frac{d}{p'}.$$

Given a set of points $E \subset B(0,1) \subset \mathbb{R}^d$ and sets of lines $F_1 \subset I_1 \times \mathbb{R}^{d-1}$ and $F_2 \subset I_2 \times \mathbb{R}^{d-1}$ consider the quantity

$$\langle F_1, F_2, E \rangle_T := \int_{\mathbb{R}^d} T^*[\chi_{F_1}](x) T^*[\chi_{F_2}](x) \chi_E(x) \, dx.$$

Our aim is to obtain restricted weak type inequalities,

$$\langle F_1, F_2, E \rangle_T \lesssim |E|^{1-2/p'} |F_1|^{1/q'} |F_2|^{1/q'}.$$

After pigeonholing and losing a $\log^2$, we can assume that $T^*[\chi_{F_1}] \approx \beta_1$ on E and $T^*[\chi_{F_2}] \approx \beta_2$ on E (this slightly changes (p', q') that we obtain at the end). Define

$$\alpha_1 := \frac{\langle F_1, F_2, E \rangle_T}{\beta_2 |F_1|} \approx \frac{\int_{\mathbb{R}^d} \chi_{F_1}(x) T(\chi_E)(x) \, dx.}{|F_1|}$$

and

$$\alpha_2 := \frac{\langle F_1, F_2, E \rangle_T}{\beta_1 |F_2|} \approx \frac{\int_{\mathbb{R}^d} \chi_{F_2}(x) T(\chi_E)(x) \, dx.}{|F_2|}.$$

Note that

$$\beta_1 \beta_2 \approx \frac{\langle F_1, F_2, E \rangle_T}{|E|},$$

and, on average, $T\chi_E \approx \alpha_j$ on F_j. We restrict ourself to the case when $d = 2D$ is even. Define

$$\phi_1(u_1) = (u_1, x_0 - t_0 \theta(u_1)),$$
$$\phi_2(u_1, t_1) = x_0 - t_0 \theta(u_1) + t_1(\theta(u_1) + e_d),$$
$$\ldots$$

$$\phi_{d-1}(u_1, t_1, \ldots, t_{D-1}, u_D) = \left(u_D, x_0 - t_0 \theta(u_1) + \sum_{j=1}^{D-1} t_j[\theta(u_j) - \theta(u_{j+1})] \right),$$

$$\phi_d(u_1, t_1, \ldots, u_D, t_D) = x_0 - t_0 \theta(u_1) + \sum_{j=1}^{D-1} t_j[\theta(u_j) - \theta(u_{j+1})] + t_D(\theta(u_D) + e_d).$$

We can set up a parameter space tower Ω with $|\Omega| = (\beta_1 \alpha_1)^{D - \lfloor D/2 \rfloor} (\beta_2 \alpha_2)^{\lfloor D/2 \rfloor}$ so that

$$\phi_j \to F_1, \quad \text{if } j = 1 \pmod 4,$$
$$\phi_j \to F_2, \quad \text{if } j = 3 \pmod 4,$$
$$\phi_j \to E, \quad \text{if } j \text{ is even.}$$

Then the Jacobian of ϕ_d, [9], is

$$|J| = c_d \prod_{j=1}^{D} |t_j - t_{j-1}| \prod_{1 \le j < k \le D} |u_j - u_k|^4.$$

Let $p(j) := 1$ if j is odd, and $p(j) := 2$ if j is even. By the transversality $|u_j - u_k| \gtrsim 1$ if $p(j) \ne p(k)$. By refining Ω, we can assume that $|u_j - u_k| \gtrsim \beta_{p(k)}$ if $p(j) = p(k)$, and $|t_j - t_{j-1}| \gtrsim \alpha_{p(j)}$. Thus

$$|E| \gtrsim |\Omega| \, \alpha_1^{D - \lfloor D/2 \rfloor} \alpha_2^{\lfloor D/2 \rfloor} \prod_{1 \le j < k \le D, \; p(j) = p(k)} \beta_{p(k)}^4.$$

We can symmetrize this by switching the roles of α_1, β_1 and α_2, β_2 and obtain

$$|E| \gtrsim (\alpha_1\alpha_2)^D (\beta_1\beta_2)^{D/2} \prod_{1 \leq j < k \leq D,\, p(j)=p(k)} (\beta_1\beta_2)^2.$$

This gives us

$$|E| \gtrsim (\alpha_1\alpha_2)^D (\beta_1\beta_2)^{D(D-1)/2}, \quad \text{if } D \text{ is even,}$$

$$|E| \gtrsim (\alpha_1\alpha_2)^D (\beta_1\beta_2)^{(D^2-D+1)/2}, \quad \text{if } D \text{ is odd.}$$

Plugging in the notation, we have

$$\langle F_1, F_2, E \rangle_T \lesssim |E|^{\frac{D^2-3D+2}{D^2+D}} (|F_1||F_2|)^{\frac{2}{D+1}}, \quad \text{if } D \text{ is even,}$$

$$\langle F_1, F_2, E \rangle_T \lesssim |E|^{\frac{D^2-3D+3}{D^2+D+1}} (|F_1||F_2|)^{\frac{2D}{D^2+D+1}}, \quad \text{if } D \text{ is odd.}$$

This corresponds to $p' = \frac{D+1}{2}\frac{d}{d-1}, q' = \frac{D+1}{2}$ for even D, and $p' = \frac{D^2+D+1}{d-1}, q' = \frac{D^2+D+1}{d}$ for odd D.

A.2. **Unrestricted directions.** For $z \in \mathbb{R}^{d-1}$, we have the unrestricted X-ray transform $T = T^\theta$ where $\theta(z) = z$. Perhaps the estimates of most interest are the d-linear estimates

$$\| \prod_{j=1}^{d} T^*[g_j] \|_{L^{\frac{p'}{d}}} \lesssim \prod_{j=1}^{d} \|g_j\|_{L^{q'}}.$$

where, say, each g_j is supported on $B(\xi_j, \frac{1}{100})$, and where $\xi_j = e_j$ for $j = 1, \ldots, d-1$ and $\xi_d = 0$, see [1]. We give a sketch of the method when $d = 3$ for the weaker inequality

(42) $$\left|\{x : T^*[\chi_{F_j}] \geq \beta_j, j = 1, \ldots, d\}\right| \lesssim \left(\prod_{j=1}^{d} \frac{|F_j|^{1/q'}}{\beta_j} \right)^{p'/d},$$

which follows from

$$\int \prod_{j=1}^{d} T^*[\chi_{F_j}](x)\chi_E(x)dx \lesssim |E|^{1-d/p'} \prod_{j=1}^{d} |F_j|^{1/q'}$$

where $E = \{x : T^*[\chi_{F_j}] \approx \beta_j, j = 1, \ldots, d\}$.

Consider the inflated map

$$\phi(z_1, t_{1,1}, z_{1,1}, t_{2,1}, z_{2,1})$$
$$= ((z_{1,1}, x_0 + (t_{1,1} - t_0)z_1 - t_{1,1}z_{1,1}), (z_{2,1}, x_0 + (t_{2,1} - t_0)z_1 - t_{2,1}z_{2,1}))$$

which has Jacobian

$$|J| = |(t_{1,1} - t_0)(t_{2,1} - t_0)\det(z_{1,1} - z_1, z_{2,1} - z_1)|.$$

We can construct a "Parameter space tree" $\Omega = \{(z_1, t_{1,1}, z_{1,1}, t_{2,1}, z_{2,1})\}$ of measure $\alpha_1^2 \beta_1 \beta_2 \beta_3$ so that $z_1 \in B(\xi_1, \frac{1}{100})$, $z_{1,1} \in B(\xi_2, \frac{1}{100})$, and $z_3 \in B(\xi_3, \frac{1}{100})$, and

$$\phi(\Omega) \subset F_2 \times F_3.$$

Above,

$$\alpha_1 = \frac{\int T^*[\chi_{F_1}]T^*[\chi_{F_2}]T^*[\chi_{F_3}]\chi_E}{\beta_2\beta_3|F_1|}, \text{ etc..}$$

By our "trilinear" hypotheses on $z_1, z_{1,1}$, and $z_{2,1}$, we have $|\det(z_{1,1} - z_1, z_{2,1} - z_1)| \gtrsim 1$ on Ω. Thus

$$|F_2||F_3| \gtrsim \beta_1\beta_2\beta_3\alpha_1^4.$$

Combined with the permuted estimates, we have

$$(|F_1||F_2||F_3|)^2 \gtrsim (\beta_1\beta_2\beta_3)^3(\alpha_1\alpha_2\alpha_3)^4.$$

This gives (42) for $d = 3$ with $p = \frac{7}{3}, q = 7$. These exponents are weaker than those implied by [19], but nonetheless illustrate that the method of [9] may yield estimates with unrestricted directions beyond those of [12] and [3].

A.3. **The Loomis-Whitney inequality.** For $j = 1, \ldots, d$, let $\pi_j : \mathbb{R}^d \to \mathbb{R}^{d-1}$ be the map $\pi_j(x_1, \ldots, x_d) = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots x_d)$. The Loomis-Whitney inequality [16] states that

$$\int_{\mathbb{R}^d} \prod_{j=1}^d f_j(\pi_j(x)) \, dx \leq \prod_{j=1}^d \|f_j\|_{L^{d-1}}$$

for functions $f_1, \ldots, f_d$ on $\mathbb{R}^{d-1}$.

Below, we give (except for a constant factor) a "multilinear T^*T" proof for the case $f_j = \chi_{E_j}$.

Suppose $E_1, \ldots, E_d \subset \mathbb{R}^{d-1}$ with each $|E_i| < \infty$ and let

$$\langle E_1, \ldots, E_d \rangle = \int_{\mathbb{R}^d} \prod_j \chi_{E_j}(\pi_j(x)) \, dx.$$

Define

$$\alpha_j = \frac{\langle E_1, \ldots, E_d \rangle}{|E_j|},$$

and given $y \in \mathbb{R}^{d-1}$ and a function g on $\mathbb{R}^d$ let

$$T^i[g](y) = \int_{\mathbb{R}} g(y + te_i) \, dt.$$

so that

$$\int_{\mathbb{R}^d} \prod_j f_j(\pi_j(x)) \, dx = \int_{\mathbb{R}^{d-1}} f_i(y) T^i\Big[\prod_{j \neq i} f_j(\pi_j(\cdot))\Big](y) \, dy.$$

For $i = 1, \ldots, d$ let

$$E_i' = \Big\{ y \in E_i : T^i\Big[\prod_{j<i} \chi_{E_j'(\pi_j(\cdot))} \prod_{j>i} \chi_{E_j}(\pi_j(\cdot))\Big] \geq \frac{\alpha_i}{2^i} \Big\}$$

and, by induction on i, note that

$$\langle E_1', \ldots, E_i', E_{i+1}, \ldots, E_d \rangle \geq \frac{\langle E_1, \ldots, E_d \rangle}{2^i}.$$

We may then find a parameter space tower $\Omega = \{t_d, \ldots, t_1\} \subset \mathbb{R}^d$ and $y \in E_d'$ so that $|\Omega| \geq 2^{-\frac{d(d+1)}{2}} \prod_{j=1}^d \alpha_i$ and so that

$$\pi_i(y + \sum_{j=1}^d t_j e_j) \in E_i$$

for every i and $(t_1, \ldots, t_d) \in \Omega$.

Thus

$$\langle E_1, \ldots, E_d \rangle \geq 2^{-\frac{d(d+1)}{2}} \prod_{j=1}^d \alpha_j,$$

or in other words

$$\langle E_1, \ldots, E_d \rangle \leq 2^{-\frac{d(d+1)}{2}} \prod_{j=1}^d |E_j|^{\frac{1}{d-1}}.$$

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