(1) Find the general solution for
  (a) $\frac{dy}{dt} = (1+t)(1+y)$;
  (b) $\frac{dy}{dt} = 1 - t + y^2 - ty^2$.

(2) Solve the initial value problem and determine the interval of existence for your solution:
  (a) $t^2(1+y^2) + 2y\frac{dy}{dt} = 0$, $y(0) = 1$;
  (b) $\frac{dy}{dt} = \frac{3t^2+4t+2}{2(y-1)}$, $y(0) = -1$.

(3) Consider the (not necessarily separable) differential equation

$$(*) \qquad \frac{dy}{dt} = f\left(\frac{y}{t}\right).$$

  (a) Show that the substitution $y = tv$ and some algebra transform (*) into a separable equation

$$\frac{dv}{dt} = \cdots.$$

  (b) Find the general solution to $2ty\frac{dy}{dt} = 3y^2 - t^2$.

(1) In 1880 a number of yearling bass were seined in New Jersey, taken across the continent in tanks by train, and planted in San Francisco Bay. A total of only 435 fish survived the rigors of the trip. Yet in 1900 the commercial catch alone was 1,234,000 pounds. Since the growth of this population was so fast, it is reasonable to assume that it obeyed the Malthusian law $dp/dt = ap$. Assuming that the average weight of one of these bass was 3 pounds and that in 1900 every tenth bass was caught, find a lower bound for a.

(2) Suppose that a population doubles in 100 years and triples in 200 years. Can it obey the Malthusian law?

(3) For certain classes of organisms the birth rate is *not* proportional to the population size. Suppose, for example, that each member of the population requires a partner for reproduction and that each member relies on chance encounters for meeting a mate. If the expected number of encounters is proportional to the product of the numbers of males and females, and if these are equally distributed in the population, then the number of encounters, and so the birth rate too, is proportional to p^2. The death rate is still proportional to p, so $dp/dt = bp^2 - ap$ for some $a, b > 0$. Show that if $p_0 < a/b$ then $p(t) \to 0$ as $t \to \infty$. That is, once the population drops below the critical size a/b, it tends to extinction. Thus a species is classified as endangered if its current size is close to its critical size.

(1) The equation $\frac{dp}{dt} = ap^{\alpha}$ is proposed as a model for population growth. Show that if $\alpha > 1$ then $p(t) \to \infty$ in finite time (so that this equation is **not** a suitable model).

(2) A 500 gallon tank originally contains 100 gallons of fresh water. Beginning at time $t = 0$, water containing 50 percent pollution flows into the tank at the rate of 2 gallons per minute. The well-stirred mixture leaves at the rate of 1 gallon per minute. Find the concentration of the pollutants in the tank at the moment it begins to overflow.

(3) The presence of toxins in a certain medium destroys a strain of bacteria at a rate jointly proportional to the number N of bacteria present at time t and the amount T of the toxin present at time t. Call the constant of proportionality a. If there were no toxins present, the bacteria would grow at a rate proportional to the number present. Call that constant of proportionality b. Assume that T is increasing at a constant rate c and that $T(0) = 0$. What happens to $N(t)$ as $t \to \infty$?

(1) Let

$$A = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$$

and find A^{-1}.

(2) Use A^{-1} to solve the system of equations

$$2x + 4y = c_1$$

$$x + 3y = c_2.$$

(3) Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

Show that $A(\lambda\vec{u} + \mu\vec{v}) = \lambda(A\vec{u}) + \mu(A\vec{v})$ for any scalars λ, μ.

(4) With A as in (1), find the scalars λ for which the matrix $A - \lambda I$ is not invertible.

YSP

Homework 4

(1) Show that the operator defined by

$$L[y](t) = \int_a^t s^2 y(s)\, ds$$

is linear.

(2)

(a) Show that $y_1(t) = \sqrt{t}$ and $y_2(t) = 1/t$ are solutions of

$$2t^2y'' + 3ty' - y = 0$$

on the interval $(0, \infty)$. Compute $W[y_1, y_2](t)$. What happens as $t \to 0$?

(b) Solve the initial value problem $2t^2y'' + 3ty' - y = 0$, $y(1) = 2$, $y'(1) = 1$.

(3) Show that $y(t) = t^2$ can never be a solution to

(*)
$$y'' + py' + qy = 0$$

if p and q are continuous at $t = 0$.

(4) Show that if $\{y_1, y_2\}$ is a fundamental set of solutions to (*), then y_1 and y_2 cannot have a common point of inflection unless both p and q vanish at that point.

(1) Let $L[y] = y'' - 2ay' + a^2y$.
   (a) Consider the differential equation $L[y] = 0$. Show that the characteristic equation yields (just) one solution to $L[y] = 0$ of the form $y_1(t) = e^{rt}$.
   (b) Let $v(t)$ be an (unknown) function, set $y_2(t) = v(t)y_1(t)$, and compute $L[y_2]$.
   (c) Determine a condition on $v(t)$ which is equivalent to $L[y_2] = 0$ and thus find another solution ($y_2(t) = v(t)y_1(t)$) to $L[y] = 0$.
   (d) Now you have two solutions y_1 and y_2 to $L[y] = 0$. Compute $W[y_1, y_2](t)$ and conclude that $c_1y_1 + c_2y_2$ is the general solution to $L[y] = 0$.

(2) Write the general solution to the D.E. and then solve the initial value problem:
   (a) $y'' - 4y' + 4y = 0$, $y(0) = 1$, $y'(0) = -1$;
   (a) $y'' - 4y' + 5y = 0$, $y(0) = 1$, $y'(0) = -1$;
   (a) $y'' - 4y' + 3y = 0$, $y(0) = 1$, $y'(0) = -1$.

(3) The equation $t^2y'' + \alpha ty' + \beta y = 0$ is called Euler's equation. The presence of t and t^2 suggests the ansatz $y = t^r$. What can you do with this?

YSP Homework 7

Let $L[y] = 2y'' + 3y' + y$.

(1) Find particular solutions of $L[y] = t^2$ and $L[y] = \sin t$.

(2) Find the general solution of $L[y] = 3t^2 - 2\sin t$.

(3) Find the solution of the initial value problem $L[y] = 3t^2 - 2\sin t$, $y(0) = 2$, $y'(0) = -1$.

Educated Guesses for Certain Nonhomogeneous D.E.'s

To solve $ay'' + by' + cy = g(t)$:

(1) if $g(t)$ is a polynomial P_n of degree n, try $Y(t) = t^s(A_0 + A_1t + A_2t^2 + \cdots + A_nt^n)$, where s is the number of times 0 is a root of the characteristic equation $ar^2 + br + c = 0$;

(2) if $g(t) = P_n(t)e^{\alpha t}$, try $Y(t) = t^s(A_0 + A_1t + A_2t^2 + \cdots + A_nt^n)e^{\alpha t}$, where s is the number of times α is a root of the characteristic equation;

(2) if $g(t) = P_n(t)e^{\alpha t}\big(a_1\sin(\beta t) + a_2\cos(\beta t)\big)$, try

$$Y(t) = t^s(A_0 + A_1t + A_2t^2 + \cdots + A_nt^n)e^{\alpha t}\sin(\beta t) + t^s(B_0 + B_1t + B_2t^2 + \cdots + B_nt^n)e^{\alpha t}\cos(\beta t)$$

where s is the number of times $\alpha + i\beta$ is a root of the characteristic equation.

(1) Suppose $\{a_n\}$ and $\{b_n\}$ are sequences such that for some $N_0 \in \mathbf{N}$ we have $a_n \leq b_n$ whenever $n \geq N_0$. Suppose that $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$. Show that then $L \leq M$. (Hint: assume that $M < L$, let $2\epsilon = L - M$, and derive a contradiction.)

(2) Show, according to the definition, that $\frac{1}{1+x^2}$ is continuous on $\mathbf{R}$. (Hint: show that $|x+y| \leq (1+x^2)(1+y^2)$.)

(3) Suppose $f(x)$ is continuous on $[0,1]$ and $0 \leq f(x) \leq 1$ for all $x \in [0,1]$. Show that there is some $x_0 \in [0,1]$ with $f(x_0) = x_0$. (Hint: look at $g(x) = f(x) - x$.)

1. Suppose that $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$. Show that $\lim_{n\to\infty}(a_n \cdot b_n) = L \cdot M$.

2. Suppose the sequence $\{s_n\}$ is defined recursively by the formula $s_{n+1} = \sqrt{2+\sqrt{s_n}}$ and the initial condition $s_1 = \sqrt{2}$. Show that $\{s_n\}$ converges to some number s. For extra credit, argue that s is actually an algebraic number. (You will probably need to look up the definition of algebraic number.)

(1) Find eigenvalues and eigenvectors for:

(a)
$$\begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}$$

(b)
$$\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$$

(c)
$$\begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$$

(you may need complex numbers for this one).

(2) Find equilibrium solutions to:

(a)
$$\frac{dx}{dt} = x - x^2 - 2xy, \quad \frac{dy}{dt} = 2y - 2y^2 - 3xy$$

(b)
$$\frac{dx}{dt} = xy^2 - x, \quad \frac{dy}{dt} = x\sin(\pi y)$$

(c)
$$\frac{dx}{dt} = -\beta xy + \mu, \quad \frac{dy}{dt} = \beta xy - \gamma y.$$

YSP Homework 10

Consider L.F. Richardson's equations

(1) $$\frac{dx_1}{dt} = kx_2 - \alpha x_1 + g, \ \frac{dx_2}{dt} = lx_1 - \beta x_2 + h.$$

(a) Using eigenvalues derive the solutions

$$x_1(t) = Ae^{\sqrt{kl}\,t} + Be^{-\sqrt{kl}\,t}, \ x_2(t) = \sqrt{\frac{l}{k}}\left[Ae^{\sqrt{kl}\,t} - Be^{-\sqrt{kl}\,t}\right]$$

in the "arm's race" case $\alpha = \beta = g = h = 0$.

(b) Derive the condition $\alpha\beta > kl$ for stability of the equilibrium solution to (1) and interpret it.

(1) Find the equilibrium solutions:
  (a) $\frac{dx}{dt} = x - x^2 - 2xy,\ \frac{dy}{dt} = 2y - 2y^2 - 3xy;$
  (b) $\frac{dx}{dt} = xy^2 - x,\ \frac{dy}{dt} = x\sin(\pi y);$
  (c) $\frac{dx}{dt} = -\beta xy + \mu,\ \frac{dy}{dt} = \beta xy - \gamma y.$

(2) Find the orbits and determine the "direction of travel":
  (a) $\frac{dx}{dt} = y + x^2 y,\ \frac{dy}{dt} = 3x + xy^2;$
  (b) $\frac{dx}{dt} = xye^{-3x},\ \frac{dy}{dt} = -2xy^2;$
  (b) $\frac{dx}{dt} = 4y,\ \frac{dy}{dt} = x + xy^2.$

(3) Show that any solution of $\frac{dx}{dt} = y(e^x - 1),\ \frac{dy}{dt} = x + e^y$ which starts in the right half plane ($x > 0$) stays there for all time.

YSP Homework 12

Consider the system of equations

$$(*) \qquad \frac{dN_1}{dt} = a_1 N_1(K_1 - N_1 - \alpha N_2), \quad \frac{dN_2}{dt} = a_2 N_2(K_2 - N_2 - \beta N_1).$$

(1) Assume that $K_1/\alpha > K_2$ and $K_2/\beta < K_1$. Show $N_2(t) \to 0$ as $t \to \infty$ so long as $N_1(t_0) > 0$ for some t_0.

(2) Assume that $K_1/\alpha > K_2$ and $K_2/\beta > K_1$. Show $N_1(t) \to (K_1 - \alpha K_2)/(1 - \alpha\beta)$ and $N_2(t) \to (K_2 - \beta K_1)/(1 - \alpha\beta)$ as $t \to \infty$ so long as $N_1(t_0), N_2(t_0) > 0$ for some t_0.