Divergence in Coxeter groups

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Joint work with

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Geometry of groups

Let G be a finitely presented group: $\quad G = \langle A \mid R \rangle$

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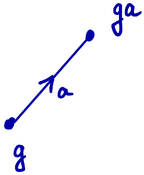
- Cayley graph: $\mathrm{Cay}^1_A(G)$:
  - $\{\text{vertices}\} \longleftrightarrow G$
  - $\{\text{directed edges}\} \longleftrightarrow G \times A$



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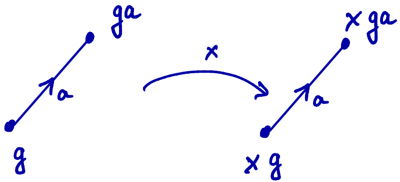

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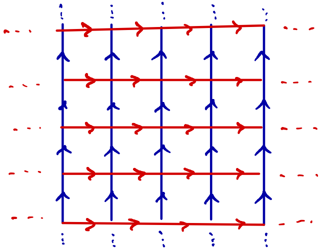
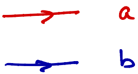
G acts on $\mathrm{Cay}^1_A(G)$ and $\mathrm{Cay}^2_{\langle A|R \rangle}(G)$ by isometries.

Examples:

$\mathbb{Z} \times \mathbb{Z} = \langle a, b \mid [a, b] \rangle$:





$\mathrm{Cay}^2_{\langle A|R \rangle} = \mathbb{R}^2$

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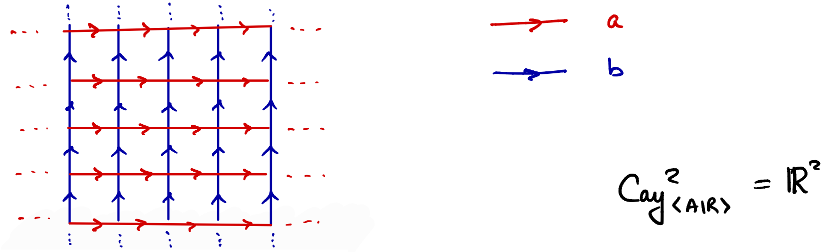
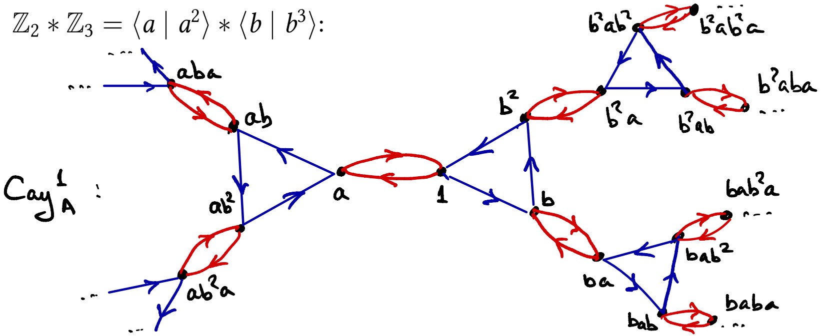


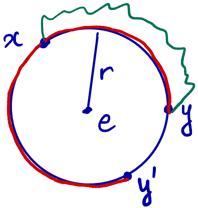
$\mathbb{Z}_2 * \mathbb{Z}_3 = \langle a \mid a^2 \rangle * \langle b \mid b^3 \rangle$:



Divergence

Let

- X be a 1-ended geodesic metric space
- e basepoint,
- $S(e, r)$ sphere of radius r around e



The **divergence** of X is:

$$\mathrm{div}_X(r) = \sup_{x,y \in S(e,r)} \inf\big(\text{lengths of } r\text{-avoidant paths from } x \text{ to } y\big)$$

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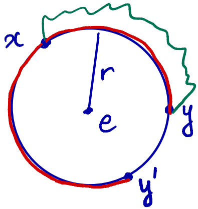

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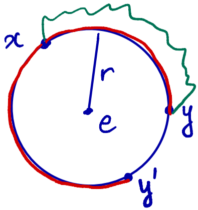

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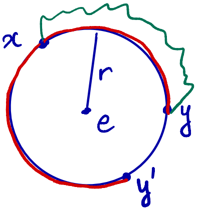

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Examples

- $\mathbb{R}^2$: $\operatorname{div}_{\mathbb{R}^2}(r) = \pi r$, linear
- $\mathbb{H}^2$: $\operatorname{div}_{\mathbb{H}^2}(r) = \pi \sinh(r) \sim \pi e^r/2$, exponential

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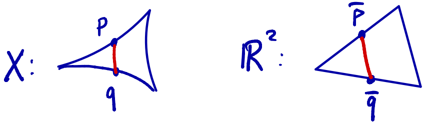

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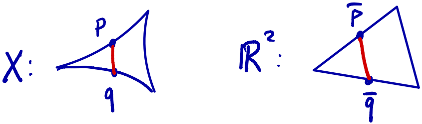

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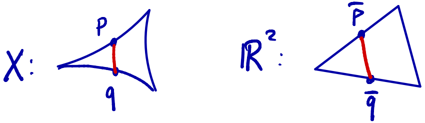

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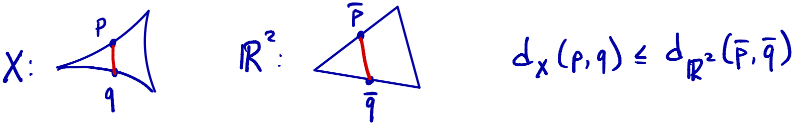

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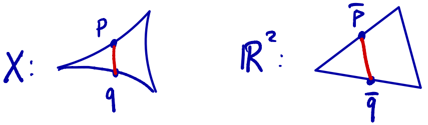

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$\Big\}\ F_n \rtimes_\varphi \mathbb{Z}$, $\varphi \in \mathrm{Aut}(F_n)$

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Q: Given your favorite class of groups, what spectrum of divergence functions does it have?

Coxeter groups

A Coxeter group W is given by:

- finite set S
- symmetric matrix $(m_{st})_{s,t \in S}$ such that:

$$m_{ss} = 1,\ m_{st} = m_{ts} \in \{2, 3, 4, \ldots, \infty\}$$

(W, S) is given by presentation:

$m_{ss} = 1$

$$W = \langle S \mid (st)^{m_{st}} = 1,\ \text{for all } s, t \in S \rangle$$

$(ss)^1 = 1$

$m_{st} = \infty$ means that st has infinite order.

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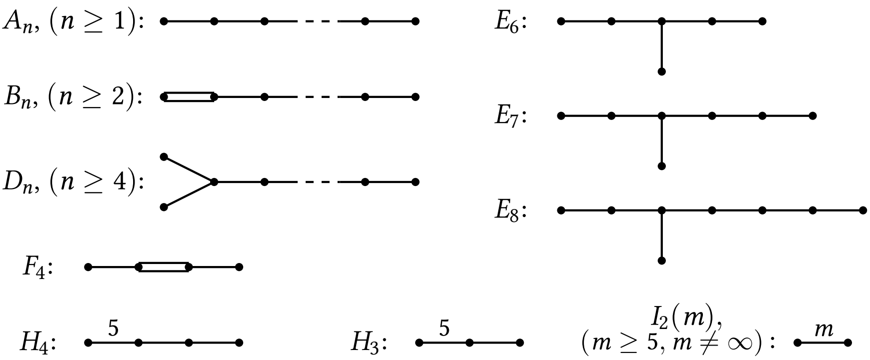
Encoded by a Coxeter graph (a.k.a. Dynkin graph) with edges lableled m_{st}:

Spherical Coxeter groups = finite

Here's the list of irreducible ones (Coxeter, 1935):



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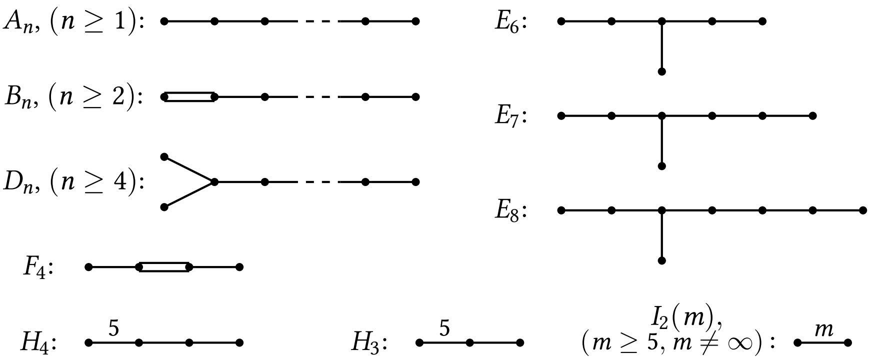
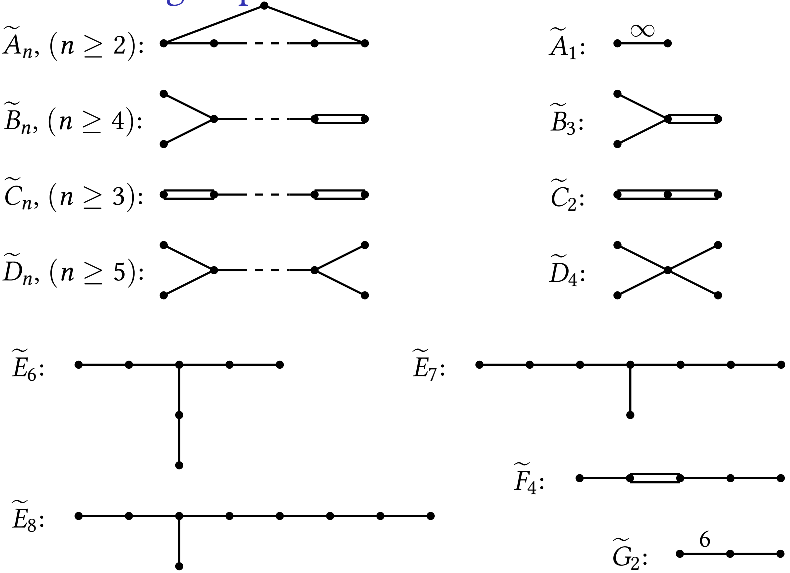


Groups generated by reflections in
the faces of a simplex in $\mathbb{S}^n$

Affine Coxeter groups



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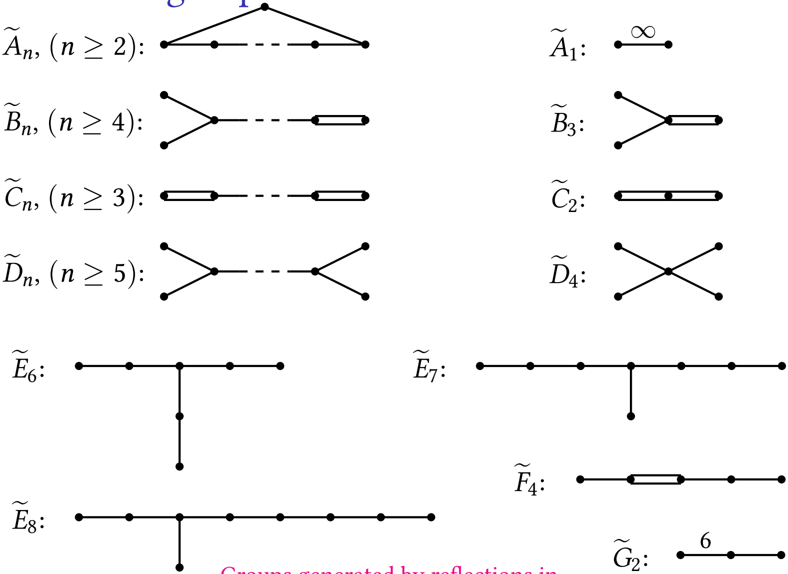

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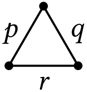
Lannér's hyperbolic Coxeter groups



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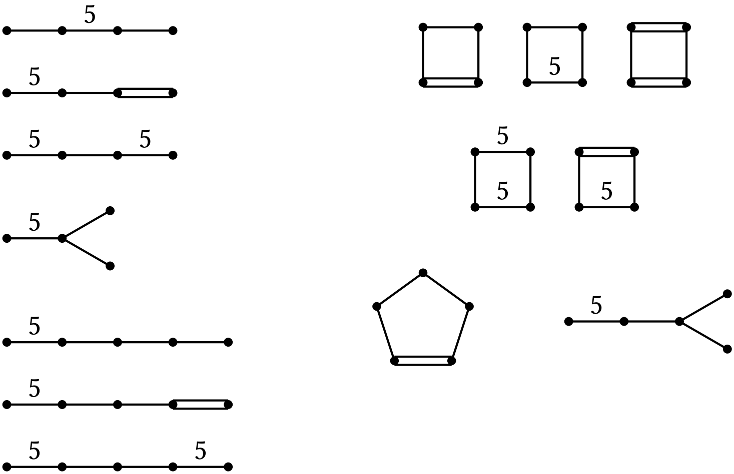

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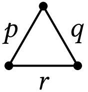

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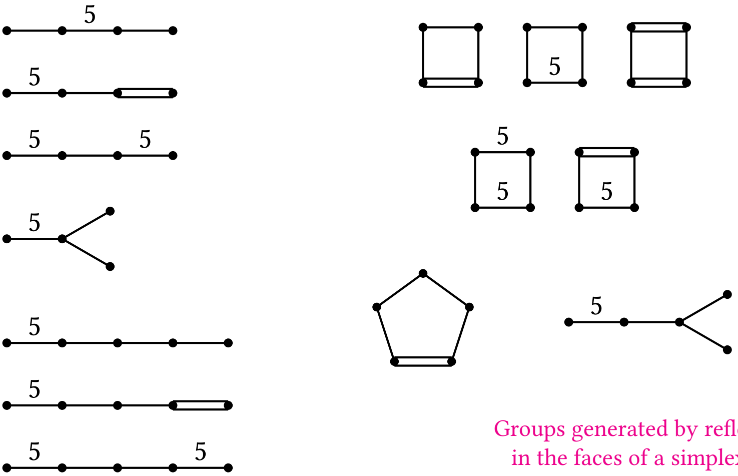

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Our results, part I

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As a corollary we get a complete characterization of linear divergence:

Corollary 2

Let (W, S) be a 1-ended Coxeter system. Then W has linear divergence if and only if $(W, S) = (W_1, S_1) \times (W_2, S_2)$ where either

1. *both W_1 and W_2 are infinite, or* $\to \mathbb{Z}\times\mathbb{Z}$
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Corollary 3

If a 1-ended Coxeter group has a super-linear divergence, then its divergence is at least quadratic.

I.e. there is a gap between r and r^2.

similar is true for Dehn functions

Our results, part II

$m_{st} = 2, \infty$

Ivan Levcovitz introduced what he called a hypergraph index for RACGs, which is an integer ≥ 0 or ∞, computable directly from the Coxeter graph.

We generalize it for general Coxeter groups.

Theorem 4

1. $h = 0 \iff$ *W has linear divergence.*
2. $h = 1 \implies$ *W has quadratic divergence.*
3. *h is finite $\implies$ the divergence of W is bounded above by a polynomial of degree $h + 1$.*
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Levcovitz (2020): true for **right-angled** Coxeter groups ($m_{s,t} \in \{2, \infty\}$).

We proved it for certain series of non-right-angled Coxeter groups.

Our results, part III

Theorem 5

Let (W, S) *be a Coxeter system with the Coxeter graph* $\Delta = \Delta(W, S)$ *and hypergraph index* $h = h(W, S)$*. If* h *is finite then* $h \leq \mathrm{b}_1(\Delta) + 1$*, where* $\mathrm{b}_1(\Delta)$ *is the 1-st Betti number of* Δ*.*

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$\mathrm{b}_1(\Delta) = e - v + k$, $\quad v = \#\text{vertices}$, $\quad e = \#\text{edges}$, $\quad k = \#\text{components}$.

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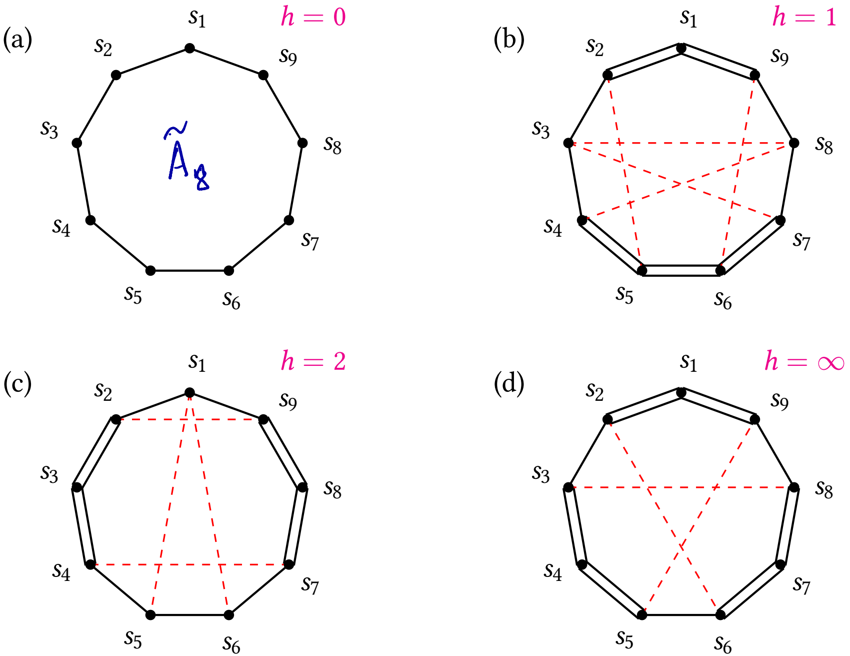
$$\mathrm{b}_1(\Delta) = e - v + k, \quad v = \#\text{vertices}, \quad e = \#\text{edges}, \quad k = \#\text{components}.$$

Corollary 6

If a Coxeter group W *is not relatively hyperbolic, then the divergence of* W *is bounded above by a polynomial of degree* $\mathrm{b}_1(\Delta) + 2$*.*

Corollary 7

If the Coxeter graph of (W, S) *is a tree and* W *is* 1*-ended, then* W *has divergence linear, quadratic or exponential only. Moreover, each of these possibilities is realized.*



(a)
$h = 0$
s_1 s_2 s_3 s_4 s_5 s_6 s_7 s_8 s_9
$\tilde{A}_8$
(b)
$h = 1$
s_1 s_2 s_3 s_4 s_5 s_6 s_7 s_8 s_9
(c)
$h = 2$
s_1 s_2 s_3 s_4 s_5 s_6 s_7 s_8 s_9
(d)
$h = \infty$
s_1 s_2 s_3 s_4 s_5 s_6 s_7 s_8 s_9

Key idea

Behrstock–Caprace–Hagen–Sisto (2017): A Coxeter group W is either:

- relatively hyperbolic $\implies$ $\text{div} \simeq$ exponential
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Goal: Determine the exact upper bound.

A relatively hyperbolic group H has a family of peripheral subgroups P_i:

(1) Each $\mathbb{Z} \times \mathbb{Z}$ subgroup of H must be contained in some of P_i

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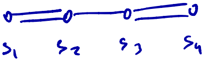
Plan: Build candidates for peripheral subgroups $P_i \subseteq W$ forced by (1) and (2). Start with obvious subgroups containing $\mathbb{Z} \times \mathbb{Z}$ and take their joins if they intersect infinitely. Once the process stops:

- if no subgroup P_i equal W: we get an honest peripheral structure and W is relatively hyperbolic $\implies$ div $\simeq$ exponential
- if some $P_i = W$, the Coxeter group W is thick, and the number of steps before stabilization is our hypergraph index $h \implies$ div $\preceq r^{h+1}$.

More formally:

S : set of gens



Wide subsets: $\Omega(S)$ = maximal sets of the form $A \times B$ where

- A, B both nonspherical, or
- A irreducible affine of $\text{rk} \geq 3$, B spherical (or empty)

} obviously contain $\mathbb{Z} \times \mathbb{Z}$

Slab subsets: $\Psi(S)$ = maximal sets of the form: $A \times K$, such that

- A is minimal nonspherical
- K is maximal nonempty spherical, commuting with A
- there does not exist $T \in \Omega(S)$ such that $A \times K \subseteq T$.

} a tool to detect when two peripheral intersect in an infinite subgroup

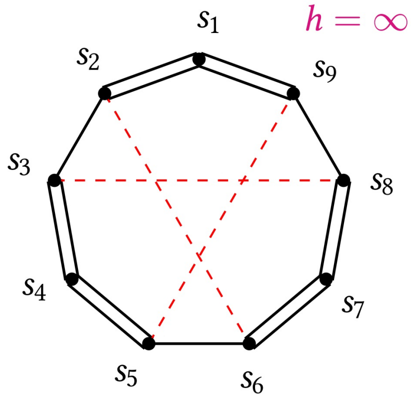
Define: $\Lambda_0(S) = \Omega(S) \cup \Psi(S)$,

$\Lambda_{i+1}(S)$ = set of all unions of elements in $\equiv_i$ equivalence class on $\Lambda_i(S)$, generated by the condition "$T \cap T'$ is nonspherical"

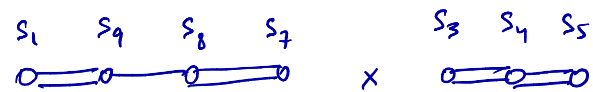
Then the hypergraph index h is:

- if $S \in \Lambda_h(S) \setminus \Lambda_{h-1}(S)$ and $\Omega(S) \neq \varnothing$: $h \in \mathbb{N}$
- otherwise $h = \infty$

(d)

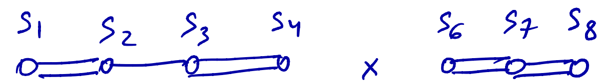


T_1:

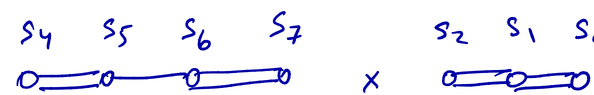


$\tilde{C}_3 \times \tilde{C}_2$

T_2:



T_3:



$\Lambda_0 = \{T_1, T_2, T_3\}$

$T_i \cap T_j =$

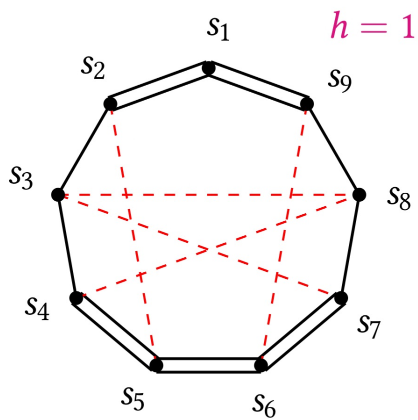
, spherical

$C_2 \times C_2 \quad A_1$ i.e. finite!

$\Lambda_1 = \Lambda_0$, relatively hyperbolic with peripheral subgroups $\{W_{T_1}, W_{T_2}, W_{T_3}\}$

$h = \infty$

(b)



$h = 1$

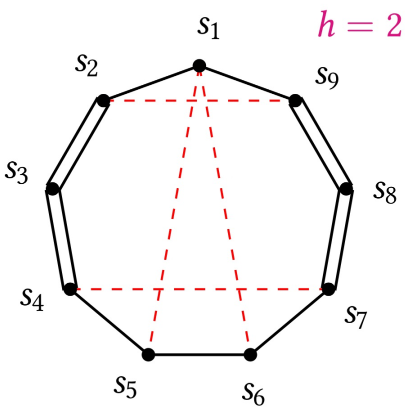
T_1: $s_2 \; s_1 \; s_9 \; s_8 \times s_4 \; s_5 \; s_6 = S \setminus \{s_3, s_7\}$

T_2: $s_3 \; s_2 \; s_1 \; s_9 \times s_5 \; s_6 \; s_7 = S \setminus \{s_4, s_8\}$

$T_1 \cap T_2$: $s_1 \; s_2 \; s_3 \times s_5 \; s_6 = \tilde{C}_2 \times B_2$

$T_1 \cup T_2$: all of S $\quad h = 1$

(c)



$h = 2$

$T_1 = S \setminus \{s_1, s_5\} = s_2 \; s_3 \; s_4 \times s_6 \; s_7 \; s_8 \; s_9$

$T_2 = S \setminus \{s_1, s_6\} = s_2 \; s_3 \; s_4 \; s_5 \times s_7 \; s_8 \; s_9$

$T_1 \cap T_2$: $s_2 \; s_3 \; s_4 \times s_7 \; s_8 \; s_9 = \tilde{C}_2 \times \tilde{C}_2$, nonspherical

$T_4 := T_1 \cup T_2 = S \setminus \{s_1\}$

$T_3 = S \setminus \{s_2, s_9\} = s_3 \; s_4 \; s_5 \; s_6 \; s_7 \; s_8 \times s_1 = \tilde{C}_5 \times A_1$ $\quad h = 2$

$T_3 \cap T_4 = \tilde{C}_5$, nonspherical, $T_5 := T_3 \cup T_4 = S$.