Property R_∞ for Artin groups

Ignat Soroko

University of North Texas

ignat.soroko@unt.edu

Joint work with

Matthieu Calvez, IRISIB, Brussels, Belgium

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Property R_∞

Let G be a group, and $\varphi \in \mathrm{Aut}(G)$. Consider an equivalence relation on G:

$$x \sim_\varphi y \quad \Longleftrightarrow \quad x = g \cdot y \cdot \varphi(g)^{-1} \quad \text{for some } g \in G$$

The equivalence class $[x]_\varphi$ of $x \in G$ is called **the Reidemeister class** of x.

The **Reidemeister number** $R(\varphi)$ of φ is the number of Reidemeister classes for all $x \in G$:

$$R(\varphi) = \mathrm{Card}\{\, [x]_\varphi \mid x \in G \,\}$$

A group G has **property R_∞** if for each automorphism $\varphi \in G$, we have

$$R(\varphi) = \infty.$$

Motivation from group theory

An old question in group theory:

Q: Does every infinite group have an infinite number of (usual) conjugacy classes? I.e. is $R(\mathrm{id}_G) = \infty$ always? The answer is **no**:

- **Higman–Neumann–Neumann (1949)**: an infinitely generated group with finite number of conjugacy classes, i.e. $R(\mathrm{id}_G) < \infty$.
- **S. Ivanov (1994)**: a finitely generated group with $R(\mathrm{id}_G) < \infty$.
- **D. Osin (2004)**: a finitely generated group with any two nontrivial elements conjugate, i.e. $R(\mathrm{id}_G) = 2$.

Property R_∞, i.e. the infiniteness of $R(\varphi)$, is a generalization of this question from $\varphi = \mathrm{id}_G$ to all automorphisms φ of G.

Motivation from fixed point theory

Let X be a compact connected polyhedron and $f\colon X \to X$ a self-map.

The **Nielsen number** $N(f)$ of f is the number of *essential* fixed point *classes* of f. Properties of $N(f)$:

- (lower bound) $0 \leq N(f) \leq \#\,\mathrm{Fix}(f)$
- (homotopy invariance) if $f \simeq h$ then $N(f) = N(h)$
- (bound realized) if X is a compact triangulable manifold of $\dim \geq 3$, then $\boxed{N(f) = \min\{\#\,\mathrm{Fix}(h) \mid h \simeq f\}}$

The **Reidemeister number** $R(\varphi)$ is the number of fixed point classes of any map $f\colon X \to X$ with the induced homomorphism $f_* = \varphi$ on $\pi_1(X)$.

Always: $N(f) \leq R(f_*)$

Often: (nil-manifolds, Jiang spaces) $\boxed{R(f_*) = \infty \quad \Longrightarrow \quad N(f) = 0}$

Examples

Examples of R_∞ groups:

- Gromov-hyperbolic (non-elementary), relatively hyperbolic groups
- Baumslag–Solitar groups (except for $\mathbb{Z} \times \mathbb{Z}$).
- Mapping class groups of surfaces (of big enough complexity)
- Weakly branch groups (Grigorchuk group, Gupta–Sidki group)
- Irreducible lattices in conn. s-simple Lie groups of real rank ≥ 2
- Free nilpotent group $N_{r,c}$ of rank r and class c iff $c \geq 2r$
- (2021) Pure Artin braid groups

Examples of groups **NOT** having property R_∞:

- free abelian groups
- free nilpotent group $N_{r,c}$ of rank r and nilpotency class $c < 2r$
- lamplighter groups $\mathbb{Z}_n \wr \mathbb{Z}$ if $\gcd(n, 6) = 1$.

Attractive conjectures:

- [?] Felshtyn–Hill (1994): A fin. gen. torsion-free group with exponential growth has property R_∞. Disproved by Gonçalves–Wong (2003).
- [?] Felshtyn–Troitsky (2012): A fin. gen. residually finite non-amenable group has property R_∞. Claimed as proved, but a gap was found. Still open!
- [?] Felshtyn–Troitsky (2015): A fin. gen. residually finite group either has property R_∞ or is solvable-by-finite.

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- [?] Felshtyn–Troitsky (2015): A fin. gen. residually finite group either has property R_∞ or is solvable-by-finite. BREAKING NEWS: Disproved by Troitsky on January 29, 2023.

Moral: Property R_∞ is a subtle thing, and finding more groups having property R_∞ poses a significant interest.

Question: Given your favorite class of groups, which groups of this class have property R_∞?

Artin groups

- S finite set of generators
- $M = (m_{s,t})_{s,t \in S}$ a symmetric matrix of $\{1, 2, 3, \ldots, \infty\}$; $m_{s,s} = 1$.

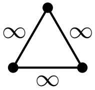
The **Artin group** corresponding to (S, M) is given by the presentation:

$$A = \left\langle \; S \;\middle|\; \underbrace{stst\ldots}_{m_{s,t}} = \underbrace{tsts\ldots}_{m_{s,t}}, \quad \forall s, t \in S,\; m_{s,t} \neq \infty \; \right\rangle$$

encoded by a graph (the Coxeter graph) with edges labeled by $m_{s,t}$:

Convention: $m_{s,t} = 2$: no edge, $m_{s,t} = 3$: label omitted

Examples:  braid group

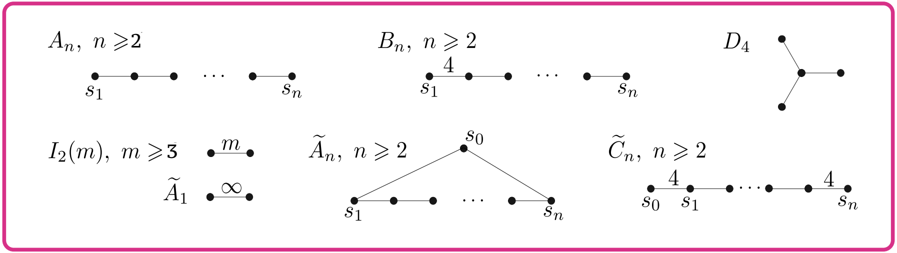


free group F_3

Our results

Theorem (Calvez–Soroko (2022))

The following Artin groups and their pure subgroups have property R_∞:



Key idea: Realize these Artin groups as f.i. subgroups in m.c.g. of punctured surfaces, and use geometry of the curve complex.

More detail:

1. Realize groups in question as subgroups of finite index in mapping class groups of punctured surfaces (by works of Charney–Crisp and Soroko):
   - sphere with $n+2$ punctures for types $A_n, B_n, \widetilde{A}_{n-1}, \widetilde{C}_{n-1}$;
   - torus with 3 punctures for D_4.
2. Using Ivanov–Korkmaz theorem observe that the same will hold for their groups of automorphisms.
3. Use the fact that mapping class groups act non-elementarily on the curve complex of the corresponding surface, which is a Gromov-hyperbolic space by results of Masur–Minsky.
4. After that a theorem of Delzant allows us to distinguish infinitely many twisted conjugacy classes.

References:

M. Calvez, I. Soroko, *Property R_∞ for some spherical and affine Artin-Tits groups.* ***Journal of Group Theory*** 25, 6 (2022), 1045–1054.