sample test 2, this test, and the actual test will be OPEN BOOK, calculators will not be allowed.

1. Find the general solution of the given differential equation:
   \[ y'' + 2y' = 3 + 4\sin(2t) \]

2. If \( p \) and \( q \) are constants, then
   \[ t^2y'' + pyt' + qy = 0 \quad t > 0, \tag{1} \]
   is called an Euler ODE.

   a. Show that Euler’s ODE has the solution \( y = t^r \) if \( r \) solves the quadratic equation \( Q(r) = r^2 + (p - 1)r + q = 0 \). Show that \( y_1 = t^{r_1} \) and \( y_2 = t^{r_2} \) form a fundamental solution set if \( r_1 \) and \( r_2 \) are distinct roots of \( Q(r) \). If \( r = \alpha + i\beta \) is a complex root, show that \( y_1 = t^\alpha \cos(\beta \ln t) \), and \( y_2 = t^\alpha \sin(\beta \ln t) \) form a fundamental solution set for the Euler ODE. (hint: write \( t^{\alpha + i\beta} \) as \( e^{(\alpha + i\beta) \ln t} \))

   b. Show that the change of independent variable \( t = e^s \) converts the Euler equation (1) into the constant coefficient ODE
   \[ \frac{d^2y}{ds^2} + (p - 1)\frac{dy}{ds} + qy = 0. \]

   c. Use the techniques presented in part (a) to solve the initial value problem
   \[ t^2y'' - 2ty' + 2y = 0 \quad y(1) = 0 \quad y'(1) = -1. \]
   What is the largest interval on which this solution is defined?

   d. Use the technique in part (b) to solve
   \[ t^2y'' + 2ty' + 2y = 0 \quad y(1) = 0 \quad y'(1) = 1. \]

3. Show that the solution \( y = Ct^2 \) solves the following initial value problem for any \( C \):
   \[ t^2y'' - 2ty' + 2y = 0 \quad y(0) = 0 \quad y'(0) = 0. \]

   Shouldn’t the fact that there are infinitely many solutions to the above initial value problem violate the uniqueness part of theorem 3.2.1. Why? Why not?
4. a. Use Abel's formula to show that if $y_1(t) \neq 0$ is a solution of

$$y'' + p(t)y' + q(t)y = 0,$$

then a second solution $v(t)$ of the ODE can be found by solving the following first order linear ODE for $v$,

$$y_1(t)v' - y_1'(t)v = e^{-\int p(t)dt}.$$

Show that the two solutions $y_1(t)$ and $v(t)$ form a fundamental set of solutions for the given ODE.

b. Given that $y_1(t) = e^t$ is a solution of

$$ty'' - (t+2)y' + 2y = 0,$$

find a second solution $v$ such that $e^t$ and $v$ is a fundamental set.