An Improved Sharp Interface Method for Viscoelastic and Viscous Two-Phase Flows

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Abstract We introduce a robust method for computing viscous and viscoelastic two-phase bubble and drop motions. Our method utilizes a coupled level-set and volume-of-fluid technique for updating and representing the air-water interface. Our method introduces a novel approach for treating the viscous coupling terms at the air-water interface; these improvements result in improved stability for computing two-phase bubble formation solutions. We also present an improved, "positive-preserving" discretization technique for updating the configuration tensor for viscoelastic flows, in the context of computing two-phase bubble and drop motion.

Keywords Level-set method · Volume-of-fluid method · Two-phase flow · Bubbles · Drops

1 Introduction

In [1, 2], and [3], a sharp interface method was developed for computing solutions to problems in Newtonian two-phase flows and viscoelastic two-phase flows. The interface separating the two phases in these papers was represented and updated using the coupled level set and volume-of-fluid method. The treatment of interfacial boundary conditions for our sharp interface method has the property that in the limit of zero gas density and zero gas viscosity, i.e. in the limit that the gas is treated as a zero density void, the solutions approach the solutions of the corresponding *one-phase* problem. The air-water interface for the *one-phase* problem is a free-surface in which the liquid on one side is treated as an incompressible fluid and the gas pressure on the other side of the free-surface is treated as spatially constant.

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We remark that Kang et al. [4] used the Ghost Fluid Method [5] and techniques in [6] to treat multiphase incompressible flow, including the effects of viscosity, surface tension, and gravity using a sharp interface method. Their method takes the interfacial jump conditions into consideration, but they use an explicit formulation of viscous terms. References [1–3], and this paper utilize an implicit treatment of the viscous terms. Our treatment takes the location of the interface into consideration and uses the knowledge that the two-phase algorithm reduces to a one-phase algorithm in the limit that gas becomes a constant pressure void.

In this paper we present two improvements, in the context of our sharp interface method; (1) we present improvements for the discretization of the "coupling" terms that appear in the viscous force terms and (2) we present improved discretization techniques for updating the configuration tensor for viscoelastic flows.

1.1 Improvements in Calculating the Viscous Force Terms

We introduce a robust method for computing viscosity in two-phase bubble and drop motions. Sussman et al. [1] described a sharp interface method for incompressible two-phase flows. This method uses a coupled level-set and volume-of-fluid technique for updating and representing the air-water interface, and introduces a novel approach for treating the jump conditions at the air-water interface. This method yielded solutions in the zero gas density limit which were comparable in accuracy to the method in which the gas pressure was treated as spatially constant. This improvement allowed for increases in the accuracy of solutions on coarse grids.

However, the above method's matrix solver would not converge when viscous effects were dominant. The above method also was inefficient in the sense that it used a second order time discretization, even though the spatial error in discretizing the viscous force terms was only first order accurate. Li et al. [7] gave a method to decouple problematic viscous terms with a low Reynolds number. This provided a basis for Sussman and Ohta to describe improvements for calculating two-phase bubble and drop motions [3].

The new method proposed by Sussman-Ohta [3] no longer suffered from problems with large viscous terms. The decoupling of the viscous terms resulted in a diagonally dominant matrix system for the non-coupled viscous force terms; such a matrix system could be robustly solved using the conjugate gradient method or multigrid method. This new method involved a first order, backwards Euler time discretization for the non-coupled viscous terms and was comparably better than [1] when viscous effects are dominant.

However, the calculation of the viscous coefficients used for the coupling terms in [3] are not as accurate as they could be, and the method will lead to instabilities in some calculations. The analysis done in [7] only considered the temporal discretization when proving unconditional stability for their method. We show through example in this paper that the semi-implicit approach taken by [3] works well for many bubble and drop two-phase flow problems, but fails for the bubble formation problem for certain parameter regimes. Here, we shall introduce a new method to improve the evaluations of the viscous coupling-term coefficients. Our method does not lead to instabilities and is either just as accurate as the methods proposed in [1] and [3], or for some problems, measurably more accurate.

Other methods of interest include [8] and [9]. In [8], Hong et al. treats their viscous terms implicitly in a method similar to that described in [3]. However, they assume that the net result of their viscous coupling terms is zero. In [9], Rasmussen et al. also use a method similar to that described in [3]. However, their method for discretizing the viscous force terms is developed in the context of one-phase flow with variable liquid viscosity, instead

of two-phase flow; we focus on the treatment of the viscous force terms at a two-phase interface where *both* density and viscosity have large jumps.

1.2 A Positive Definite Preserving Numerical Scheme for Updating the Configuration Tensor for Viscoelastic Two-Phase Flow

Algorithms for viscoelastic two-phase flow have been presented by Yu et al. [10], Pillapakkam and Singh [11], and Jimenez et al. [2]. Also, algorithms for viscoelastic one-phase flow have been presented by Goktekin et al. [12], Losasso et al. [13] and Irving [14]. We remark that the previous methods [10] and [11] were developed in the context of a "smoothed" interface method, instead of the sharp interface method presented here. The work by [12–14] was developed in the context of a one-phase flow algorithm in which the gas pressure was assumed spatially constant. We note that, as remarked by Losasso et al. [13], Goktekin et al. [12] ignored the fluid rotation term since Goktekin et al. solved the following simplified equation for the elastic strain tensor ε :

$$\varepsilon_t + \vec{u} \cdot \varepsilon = (\nabla \vec{u} + (\nabla \vec{u})^T)/2 - \varepsilon_t^{\text{Plastic}}$$

In this paper, we describe improvements to the work presented by Jimenez et al. [2] for the discretization of the evolution equation for the configuration tensor *A*:

$$\frac{\partial \mathbf{A}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{A} = \mathbf{A} \cdot (\nabla \mathbf{u})^T + \nabla \mathbf{u} \cdot \mathbf{A} - \frac{f(R)}{\lambda} (\mathbf{A} - I).$$
(1.1)

For the FENE–CR model [15], the conformation of the polymer is specified in terms of the average of the dyadic product $\langle \mathbf{RR} \rangle$ of the dumbbell end-to-end vector \mathbf{R} . c, L, λ are FENE–CR model parameters. The parameter c is a measure of the concentration of dumbbells, L is the extensibility parameter which denotes the maximum average length of s polymer molecule relative to the equilibrium end-to-end dimension, λ is a characteristic relaxation time. The FENE–CR model has a constant shear viscosity and the FENE–CR model reduces to the Oldroyd-B model as $L \rightarrow \infty$. The function f(R) specifies the nonlinear spring characteristics of the viscoelastic fluid and is given by

$$f(R) = \frac{1}{1 - \text{tr}(\mathbf{A})/L^2}.$$
 (1.2)

In particular, we observe that (1.1) preserves *A* as a positive definite matrix. The discretization of (1.1) should preserve the positive definite property of *A* as well. In our previous work [2], we only preserved the diagonal entries of *A* to be positive. We remark that there are many choices of schemes for discretizing (1.1), see e.g. [16, 17] and [18], but they are complicated, and ultimately one must reduce the order of accuracy of a method to first order in order to preserve positive definiteness of *A*. We instead base our discretization on the following observation that one can rewrite (1.1) as follows,

$$A(x,t+\Delta t) = I + ((I+\Delta t\nabla U)A(x-U\Delta t,t)(I+\Delta t\nabla U)^{T}-I)e^{\frac{-f(R)}{\lambda}\Delta t} + O(\Delta t^{2}).$$
(1.3)

We show that we can robustly calculate viscoelastic bubbly flows using our new method for updating the configuration tensor together with our sharp interface approach.

We remark that a similar "matrix transformation" technique as given in (1.3) was also implemented by [14]. I.e., [14] also implemented a semi-Lagrangian advection step to

find $\varepsilon(x - U\Delta t, t)$ and then computed the similarity transformation $S\varepsilon S^T$ where $S = \frac{1}{2}(\nabla U - \nabla U^T)$. The viscoelastic model in [14] was based on the corotational Maxwell model and the viscoelastic model in this paper is based on the FENE–CR model. The numerical discretization in [14] does not preserve the positive definite property of their elastic strain tensor, whereas our numerical discretization does preserve the configuration tensor, A, as a positive definite matrix.

2 Governing Equations

The governing equations for incompressible, immiscible two-phase flow was written by Chang et al. [19] as:

$$\rho \frac{DU}{Dt} = \nabla \cdot \left(-pI + 2\mu D + \frac{\tilde{\eta}_P}{\lambda} f(R)A \right) + \rho g \hat{z} - \sigma \kappa(F) \nabla H,$$

$$\frac{\partial \mathbf{A}}{\partial t} + U \cdot \nabla \mathbf{A} = \mathbf{A} \cdot (\nabla U)^T + \nabla U \cdot \mathbf{A} - \frac{f(R)}{\lambda} (\mathbf{A} - I),$$

$$\nabla \cdot U = 0,$$

$$\frac{D\phi}{Dt} = 0, \qquad \frac{DF}{Dt} = 0,$$
(2.1)

$$\rho = \rho_L H(\phi) + \rho_G (1 - H(\phi)), \qquad \mu = \mu_L H(\phi) + \mu_G (1 - H(\phi)), \qquad \tilde{\eta}_P = \eta_P H(\phi),$$

$$H(\phi) = \begin{cases} 1, & \phi \ge 0, \\ 0, & \phi < 0, \end{cases}$$

 ϕ is a level set function which is positive in liquid and negative in gas. *F* is a volume-of-fluid function which represents the fraction of fluid that is liquid for each computational cell. Our method follows the time-splitting that was used in Sussman and Ohta [3] for decoupling the viscous terms, and also uses the previously developed CLSVOF method by Sussman and Puckett [20] by coupling the solutions of ϕ and *F*.

Other variables used are defined as: U is the velocity field, p is the pressure, $D = \frac{1}{2}(\nabla U + \nabla U^T)$ is the rate of deformation tensor, g corresponds to the acceleration due to gravity, $\mu_L(\mu_G)$ is the viscosity of liquid (gas), $\rho_L(\rho_G)$ is the density of liquid (gas), $\kappa(F) = \nabla \cdot \frac{\nabla F}{|\nabla F|}$ is the curvature, and σ is the coefficient of surface tension. The gradient terms on the right hand side of the equation for A are defined as

$$\mathbf{A} \cdot (\nabla U)^T + \nabla U \cdot \mathbf{A} = \mathbf{A}_{\alpha \gamma} \frac{\partial U_{\beta}}{\partial x_{\gamma}} + \frac{\partial U_{\alpha}}{\partial x_{\gamma}} \mathbf{A}_{\gamma \beta}.$$

3 Viscosity: An Adaptive, Sharp Interface Treatment for the Viscous Force Terms

We present a simple and robust adaptive method for computing the viscous forces as they appear in (2.1),

$$\nabla \cdot (2\mu D). \tag{3.1}$$

Our algorithm follows the same strategy as proposed by Li, Renardy, and Renardy [7] and as implemented in Sussman-Ohta [3] by discretizing the coupling terms in (3.1) explicitly and the remainder of the terms implicitly. It was shown by Li et al. [7] that the following temporal discretization for the viscous forces is stable for any time step:

$$\rho \frac{u^{**} - u^{*}}{\Delta t} = \nabla \cdot \mu \nabla u^{**} + (\mu u_{x}^{**})_{x} + (\mu v_{x}^{*})_{y} + (\mu w_{x}^{*})_{z},$$

$$\rho \frac{v^{**} - v^{*}}{\Delta t} = \nabla \cdot \mu \nabla v^{**} + (\mu v_{y}^{**})_{y} + (\mu u_{y}^{*})_{x} + (\mu w_{y}^{*})_{z},$$

$$\rho \frac{w^{**} - w^{*}}{\Delta t} = \nabla \cdot \mu \nabla w^{**} + (\mu w_{z}^{**})_{z} + (\mu u_{z}^{*})_{x} + (\mu v_{z}^{*})_{y}.$$

The non-coupling terms are discretized using standard finite volume techniques. For example, we approximate the terms $\nabla \cdot \mu \nabla u^{**} + (\mu u_x^{**})_x$ from above as,

$$\frac{2\mu_{i+1/2,j,k}(u_{i+1,j,k}-u_{i,j,k})-2\mu_{i-1/2,j,k}(u_{i,j,k}-u_{i-1,j,k})}{\Delta x^{2}} + \frac{\mu_{i,j+1/2,k}(u_{i,j+1,k}-u_{i,j,k})-\mu_{i,j-1/2,k}(u_{i,j,k}-u_{i,j-1,k})}{\Delta y^{2}} + \frac{\mu_{i,j,k+1/2}(u_{i,j,k+1}-u_{i,j,k})-\mu_{i,j,k-1/2}(u_{i,j,k}-u_{i,j,k-1})}{\Delta z^{2}}.$$

The face centered viscosity, $\mu_{i+1/2, j}$, is defined "sharply" as,

$$\mu_{i+1/2,j} = \frac{1}{\frac{\theta_{i+1/2,j}}{\mu_L} + \frac{1 - \theta_{i+1/2,j}}{\mu_G}}$$

 $\theta_{i+1/2,j}$ is the height fraction [7, 8] given by:

$$\theta_{i+1/2,j}(\phi) = \begin{cases} 1, & \phi_{i+1,j}, \phi_{i,j} \ge 0, \\ 0, & \phi_{i+1,j}, \phi_{i,j} < 0, \\ \frac{\phi_{i+1,j}^+ + \phi_{i,j}^+}{|\phi_{i+1,j}| + |\phi_{i,j}|}, & \text{otherwise.} \end{cases}$$
(3.2)

The "+" superscript stands for the positive part, i.e. $a^+ \equiv \max(a, 0)$. The combined density is also defined sharply as,

$$H(\rho) = \begin{cases} \rho^L, & \phi \ge 0, \\ \rho^G, & \phi < 0. \end{cases}$$

In Sussman and Ohta [3], the coupling terms were discretized in two dimensions as

$$\frac{\partial(\mu v_x)}{\partial y} \approx \left(\mu_{i+1/2,j+1/2}(v_x)_{i+1/2,j+1/2} - \mu_{i+1/2,j-1/2}(v_x)_{i+1/2,j-1/2} + \mu_{i-1/2,j+1/2}(v_x)_{i-1/2,j+1/2} - \mu_{i-1/2,j-1/2}(v_x)_{i-1/2,j-1/2}\right)/2\Delta y,$$

$$\frac{\partial(\mu u_y)}{\partial x} \approx \left(\mu_{i+1/2,j+1/2}(u_y)_{i+1/2,j+1/2} - \mu_{i-1/2,j+1/2}(u_y)_{i-1/2,j+1/2} + \mu_{i+1/2,j-1/2}(u_y)_{i+1/2,j-1/2} - \mu_{i-1/2,j-1/2}(u_y)_{i-1/2,j-1/2}\right)/2\Delta x.$$

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The viscosity at a node was given by

$$\mu_{i+1/2,j+1/2} = \begin{cases} \mu_L, & \theta_{i+1/2,j+1/2} = 1, \\ \mu_G, & \theta_{i+1/2,j+1/2} = 0, \\ 0, & \mu_G = 0 \text{ and } 0 < \theta_{i+1/2,j+1/2} < 1, \\ \mu_{LG}, & \text{otherwise}, \end{cases}$$

where

$$\mu_L G = \frac{\mu_G \mu_L}{\mu_G \theta_{i+1/2, j+1/2} + \mu_L (1 - \theta_{i+1/2, j+1/2})}.$$

 $\theta_{i+1/2, j+1/2}$ is a "node fraction" defined as,

$$\theta_{i+1/2,j+1/2}(\varphi) = \begin{cases} 1, & \phi_{i+1,j}, \phi_{i,j}, \phi_{i,j+1}, \phi_{i+1,j+1} \ge 0, \\ 0, & \phi_{i+1,j}, \phi_{i,j}, \phi_{i,j+1}, \phi_{i+1,j+1} < 0, \\ \theta_{ND}, & \text{otherwise.} \end{cases}$$

The "+" superscript stands for the positive part, i.e. $a^+ \equiv \max(a, 0)$, and

$$\theta_{ND} = \frac{\phi_{i+1,j}^+ + \phi_{i,j+1}^+ + \phi_{i,j}^+ + \phi_{i+1,j+1}^+}{|\phi_{i+1,j}| + |\phi_{i,j+1}| + |\phi_{i,j+1}| + |\phi_{i+1,j+1}|}.$$

The components of the parts of the deformation tensor that are handled explicitly, e.g. the coupled terms, $(u_y)_{i+1/2, j+1/2}$ in the equation for ν and $(\nu_x)_{i+1/2, j+1/2}$ in the equations for u, were calculated at nodes using standard central differencing, i.e.,

$$(u_{y})_{i+1/2,j+1/2} = \frac{u_{i+1,j+1} - u_{i+1,j} + u_{i,j+1} - u_{i,j}}{2\Delta y}.$$

The discretization of the viscous terms in [3] had the following properties:

- 1. If the gas viscosity is zero, the velocity in gas cells, $\varphi_{i,j} < 0$, is never used. This enabled our two-phase method to be equivalent to the corresponding one-phase method in the limit of zero gas density and zero gas viscosity (i.e. gas treated as vacuum with uniform pressure).
- 2. The resulting matrix system for each velocity component is written in the following form,

$$\alpha(x)p + \beta \nabla \cdot (A(x, y)\nabla p) = f(x, y).$$
(3.3)

Where (3.3) is solved for *p*. *A* is a diagonal matrix. Reference [3] solved (3.3) on an adaptive hierarchy of grids as also described in Sect. 5.

In this paper, we discretize the coupling terms differently from [3]; in three dimensions, they appear as:

$$\begin{split} \frac{\partial u}{\partial y}\Big|_{i-1/2,j,k} &= \frac{\omega_{i,j+1,k}^{i,j,k}(u_{i,j+1,k}-u_{i,j,k}) + \omega_{i,j,k}^{i,j-1,k}(u_{i,j,k}-u_{i,j-1,k})}{\Delta y(\omega_{i,j+1,k}^{i,j+1,k} + \omega_{i,j,k}^{i-1,j+1,k} + \omega_{i-1,j+1,k}^{i-1,j-1,k})} \\ &+ \frac{\omega_{i-1,j+1,k}^{i-1,j,k}(u_{i-1,j+1,k}-u_{i-1,j,k}) + \omega_{i-1,j-1,k}^{i-1,j-1,k}(u_{i-1,j,k}-u_{i-1,j-1,k})}{\Delta y(\omega_{i,j+1,k}^{i,j+1,k} + \omega_{i,j,k}^{i-1,j-1,k} + \omega_{i-1,j+1,k}^{i-1,j-1,k})}, \end{split}$$

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$$\begin{aligned} \frac{\partial u}{\partial z}\Big|_{i-1/2,j,k} &= \frac{\omega_{i,j,k+1}^{i,j,k}(u_{i,j,k+1} - u_{i,j,k}) + \omega_{i,j,k}^{i,j,k-1}(u_{i,j,k} - u_{i,j,k-1})}{\Delta z(\omega_{i,j,k+1}^{i,j,k+1} + \omega_{i-1,j,k+1}^{i-1,j,k} + \omega_{i-1,j,k+1}^{i-1,j,k-1})} \\ &+ \frac{\omega_{i-1,j,k+1}^{i-1,j,k}(u_{i-1,j,k+1} - u_{i-1,j,k}) + \omega_{i-1,j,k}^{i-1,j,k-1}(u_{i-1,j,k} - u_{i-1,j,k-1})}{\Delta z(\omega_{i,j,k+1}^{i,j,k+1} + \omega_{i,j,k}^{i,j,k-1} + \omega_{i-1,j,k+1}^{i-1,j,k} + \omega_{i-1,j,k}^{i-1,j,k-1})}, \end{aligned}$$

where,

$$\omega_{I,J,K}^{L,M,N} = \begin{cases} 1, & s\phi_{I,J,K} \ge 0 \text{ and } s\phi_{L,M,N} \ge 0, \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \\ s = \begin{cases} 1, & \phi_{i-1,j,k} \ge 0 \text{ or } \phi_{i,j,k} \ge 0, \\ -1, & \text{otherwise}, \end{cases}$$

$$\begin{split} \frac{\partial \nu}{\partial x} \bigg|_{i,j-1/2,k} &= \frac{\omega_{i+1,j,k}^{i,j,k}(\nu_{i+1,j,k} - \nu_{i,j,k}) + \omega_{i,j,k}^{i-1,j,k}(\nu_{i,j,k} - \nu_{i-1,j,k})}{\Delta x(\omega_{i+1,j,k}^{i,j,k} + \omega_{i,j,k}^{i-1,j,k} + \omega_{i+1,j-1,k}^{i,j-1,k} + \omega_{i,j-1,k}^{i-1,j-1,k})}{+ \frac{\omega_{i+1,j-1,k}^{i,j-1,k}(\nu_{i+1,j-1,k} - \nu_{i,j-1,k}) + \omega_{i,j-1,k}^{i-1,j-1,k}(\nu_{i,j-1,k} - \nu_{i-1,j-1,k})}{\Delta x(\omega_{i+1,j,k}^{i,j,k} + \omega_{i,j,k}^{i-1,j,k} + \omega_{i+1,j-1,k}^{i,j-1,k} + \omega_{i,j-1,k}^{i-1,j-1,k})}, \\ \frac{\partial \nu}{\partial z} \bigg|_{i,j-1/2,k} &= \frac{\omega_{i,j,k+1}^{i,j,k}(\nu_{i,j,k+1} - \nu_{i,j,k}) + \omega_{i,j,k}^{i,j,k-1}(\nu_{i,j,k} - \nu_{i,j,k-1})}{\Delta z(\omega_{i,j,k+1}^{i,j,k-1} + \omega_{i,j-1,k+1}^{i,j-1,k} + \omega_{i,j-1,k-1}^{i,j-1,k-1})}{\Delta z(\omega_{i,j,k+1}^{i,j,k+1} + \omega_{i,j,k}^{i,j-1,k-1} + \omega_{i,j-1,k-1}^{i,j-1,k-1})}, \end{split}$$

where,

$$\omega_{I,J,K}^{L,M,N} = \begin{cases} 1, & s\phi_{I,J,K} \ge 0 \text{ and } s\phi_{L,M,N} \ge 0, \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \\ s = \begin{cases} 1, & \phi_{i,j-1,k} \ge 0 \text{ or } \phi_{i,j,k} \ge 0, \\ -1, & \text{otherwise}, \end{cases}$$

$$\begin{split} \frac{\partial w}{\partial x} \bigg|_{i,j,k-1/2} &= \frac{\omega_{i+1,j,k}^{i,j,k}(w_{i+1,j,k} - w_{i,j,k}) + \omega_{i,j,k}^{i-1,j,k}(w_{i,j,k} - w_{i-1,j,k})}{\Delta x(\omega_{i+1,j,k}^{i,j,k} + \omega_{i,j,k}^{i-1,j,k} + \omega_{i+1,j,k-1}^{i,j,k-1} + \omega_{i,j,k-1}^{i-1,j,k-1})}{+ \frac{\omega_{i+1,j,k-1}^{i,j,k}(w_{i+1,j,k} - w_{i,j,k-1}) + \omega_{i,j,k-1}^{i-1,j,k-1}(w_{i,j,k-1} - w_{i-1,j,k-1})}{\Delta x(\omega_{i+1,j,k}^{i,j,k} + \omega_{i,j,k}^{i-1,j,k} + \omega_{i+1,j,k-1}^{i,j,k-1} + \omega_{i,j,k-1}^{i-1,j,k-1})}, \\ &\frac{\partial w}{\partial y} \bigg|_{i,j,k-1/2} = \frac{\omega_{i,j+1,k}^{i,j,k}(w_{i,j+1,k} - w_{i,j,k}) + \omega_{i,j,k}^{i,j-1,k}(w_{i,j,k} - w_{i,j-1,k})}{\Delta y(\omega_{i,j+1,k}^{i,j,k,1} + \omega_{i,j,k-1}^{i,j-1,k-1} + \omega_{i,j,k-1}^{i,j-1,k-1}(w_{i,j-1,k-1})} + \frac{\omega_{i,j+1,k-1}^{i,j,k-1}(w_{i,j+1,k-1} - w_{i,j,k-1}) + \omega_{i,j,k-1}^{i,j-1,k-1}(w_{i,j,k-1} - w_{i,j-1,k-1})}{\Delta y(\omega_{i,j+1,k}^{i,j,k,1} + \omega_{i,j,k}^{i,j-1,k} + \omega_{i,j+1,k-1}^{i,j-1,k-1} + \omega_{i,j,k-1}^{i,j-1,k-1})}, \end{split}$$

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where,

$$\omega_{I,J,K}^{L,M,N} = \begin{cases} 1, & s\phi_{I,J,K} \ge 0 \text{ and } s\phi_{L,M,N} \ge 0, \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \\ s = \begin{cases} 1, & \phi_{i,j,k-1} \ge 0 \text{ or } \phi_{i,j,k} \ge 0, \\ -1, & \text{otherwise.} \end{cases}$$

We may now write the viscous coupling terms as

$$\begin{split} \frac{\partial \left(\mu \frac{\partial u}{\partial y}\right)}{\partial x} \bigg|_{i,j,k} &\approx \frac{\mu_{i+1/2,j,k} \frac{\partial u}{\partial y} \bigg|_{i+1/2,j,k} - \mu_{i-1/2,j,k} \frac{\partial u}{\partial y} \bigg|_{i-1/2,j,k}}{\Delta x}, \\ \frac{\partial \left(\mu \frac{\partial u}{\partial z}\right)}{\partial x} \bigg|_{i,j,k} &\approx \frac{\mu_{i+1/2,j,k} \frac{\partial u}{\partial z} \bigg|_{i+1/2,j,k} - \mu_{i-1/2,j,k} \frac{\partial u}{\partial z} \bigg|_{i-1/2,j,k}}{\Delta x}, \\ \frac{\partial \left(\mu \frac{\partial v}{\partial x}\right)}{\partial y} \bigg|_{i,j,k} &\approx \frac{\mu_{i,j+1/2,k} \frac{\partial v}{\partial x} \bigg|_{i,j+1/2,k} - \mu_{i,j-1/2,k} \frac{\partial v}{\partial x} \bigg|_{i,j-1/2,k}}{\Delta y}, \\ \frac{\partial \left(\mu \frac{\partial v}{\partial z}\right)}{\partial y} \bigg|_{i,j,k} &\approx \frac{\mu_{i,j+1/2,k} \frac{\partial v}{\partial z} \bigg|_{i,j+1/2,k} - \mu_{i,j-1/2,k} \frac{\partial v}{\partial z} \bigg|_{i,j-1/2,k}}{\Delta y}, \\ \frac{\partial \left(\mu \frac{\partial w}{\partial x}\right)}{\partial z} \bigg|_{i,j,k} &\approx \frac{\mu_{i,j,k+1/2} \frac{\partial w}{\partial x} \bigg|_{i,j,k+1/2} - \mu_{i,j,k-1/2} \frac{\partial w}{\partial x} \bigg|_{i,j,k-1/2}}{\Delta z}, \\ \frac{\partial \left(\mu \frac{\partial w}{\partial y}\right)}{\partial z} \bigg|_{i,j,k} &\approx \frac{\mu_{i,j,k+1/2} \frac{\partial w}{\partial y} \bigg|_{i,j,k+1/2} - \mu_{i,j,k-1/2} \frac{\partial w}{\partial y} \bigg|_{i,j,k-1/2}}{\Delta z}. \end{split}$$

3.1 Justification for New Method for Discretizing the Viscous Coupling Terms

In Sussman and Ohta [3] the discretization of the coupling terms was done in a manner so that the gas velocity would not be included in the discretization of the viscous force terms in the case $\mu_G = 0$. This restriction sacrificed consistency in the discretization of the coupling terms near the interface. We believe that this inconsistency is the reason why we have encountered instability for some problems when treating the coupling terms explicitly. In order to illustrate the problem, and in order to illuminate the reasoning behind our solution, we refer the reader to Fig. 1. If we were to implement the method described by Sussman and Ohta [3], one would define the following nodal viscosities:

$$\boldsymbol{\mu}_{i+1/2,\,j+1/2} = 0, \qquad \boldsymbol{\mu}_{i+1/2,\,j-1/2} = \boldsymbol{\mu}^L$$

The resulting discretization for the following coupling term $\frac{\partial(\mu u_y)}{\partial x}$ would be,

$$\left(\frac{\partial(\mu u_y)}{\partial x}\right) \approx \left(\mu_{i+1/2, j-1/2}(u_y)_{i+1/2, j-1/2} + (0)(u_y)_{i+1/2, j+1/2} - \mu_{i-1/2, j-1/2}(u_y)_{i-1/2, j-1/2} - \mu_{i-1/2, j+1/2}(u_y)_{i-1/2, j+1/2}\right)/2\Delta x.$$
(3.4)

The first two terms of (3.4) combines to be an inconsistent discretization of the quantity,

$$\mu^{L} \left(\frac{\partial u}{\partial y}\right)_{i+1/2,j}.$$
(3.5)

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The level set function does not change sign between cells (i, j) and (i + 1, j) so one should consistently define all components of the rate of deformation tensor that are located at the face (i + 1/2, j). In particular the previous method is off by a factor of two. Instead we propose the following discretization for the coupling term, $\frac{\partial(\mu u_y)}{\partial x}$,

$$\frac{\frac{\partial \left(\mu \frac{\partial u}{\partial y}\right)}{\partial x}\Big|_{i,j}}{\frac{\partial \left(\mu \frac{\partial u}{\partial y}\right)_{i+1/2,j} - \mu_{i-1/2,j}\left(\frac{\partial u}{\partial y}\right)_{i-1/2,j}}{\Delta x} = \frac{\mu^{L} \frac{1}{3} \left(\frac{u_{i,j} - u_{i,j-1}}{\Delta y} + \frac{u_{i+1,j} - u_{i+1,j-1}}{\Delta y} + \frac{u_{i,j+1} - u_{i,j}}{\Delta y}\right) - \mu^{L} \left(\frac{u_{i,j+1} - u_{i,j-1} + u_{i-1,j+1} - u_{i-1,j-1}}{4\Delta y}\right)}{\Delta x}. (3.6)$$

4 Positive Preserving Update of the Configuration Tensor and the Calculation of the Viscoelastic Force Term

The equation for the configuration tensor,

$$\frac{\partial \mathbf{A}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{A} = \mathbf{A} \cdot (\nabla \mathbf{u})^T + \nabla \mathbf{u} \cdot \mathbf{A} - \frac{f(R)}{\lambda} (\mathbf{A} - I),$$

is solved in the following steps:

1.

$$A_{i,i}^* = A^n (\vec{x} - \vec{u} \Delta t).$$
(4.1)

Equation (4.1) corresponds to a semi-Lagrangian approach in which we use linear interpolation in order to evaluate $A^n(\vec{x} - \vec{u}\Delta t)$. Linear interpolation guarantees that A^* is positive definite if A^n is.

- 2. $A_{i,j}^{**} = SA_{i,j}^{*}S^{T}, S = I + \Delta t [\nabla U^{L}]_{i,j}^{n}.$ 3. $A_{i,j}^{n+1} = \frac{\Delta t \frac{f(R)}{\lambda}}{1 + \Delta t \frac{f(R)}{\lambda}}I + \frac{1}{1 + \Delta t \frac{f(R)}{\lambda}}A_{i,j}^{**}.$
- 4. Extrapolate $A_{i,j}^{n+1}$ from the liquid regions $(\phi_{i,j} \ge 0)$ into the gas regions $(\phi_{i,j} < 0)$ using "constant" extrapolation. The extrapolation is carried out by overwriting $A_{i,j}^{n+1}$ in a

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gas cell with the corresponding value from the nearest liquid cell. Constant extrapolation guarantees that the extrapolated configuration tensor remains positive definite if the original was positive definite as well.

The viscoelastic force term, $\nabla \cdot (\frac{\tilde{\eta}_P}{\lambda} f(R)A)$, is calculated in the following steps:

(a) extrapolate the tensor, $B_{i,j} = \frac{\tilde{\eta}_P}{\lambda} f(R)_{i,j} A_{i,j}$, from cell centers to cell faces. E.g.

$$B_{i+1/2,j} = \frac{(\tilde{\eta}_p)_{i+1/2,j}}{\lambda} \frac{(f(R)A)_{i+1,j} + (f(R)A)_{i,j}}{2}, \text{ where} \\ (\tilde{\eta}_P)_{i+1/2,j} = \begin{cases} \eta_P, & \phi_{i+1,j} \ge 0 \text{ and } \phi_{i,j} \ge 0, \\ 0, & \text{otherwise.} \end{cases}$$

(b) The components of the viscoelastic force vector are then written as:

$$(L_{i,j}^{\text{viscoelastic}})_{\gamma} = \frac{(B_{i+1/2,j})_{\gamma,1} - (B_{i-1/2,j})_{\gamma,1}}{\Delta x} + \frac{(B_{i,j+1/2})_{\gamma,2} - (B_{i,j-1/2})_{\gamma,2}}{\Delta y}$$

4.1 Proof of the Positive Definite Preserving Property for the Configuration Tensor Update Step

Each of the steps one thru four in Sect. 4 above preserves the configuration tensor as a positive definite symmetric matrix. This fact can be proved so long as the following stability constraint is satisfied,

$$\Delta t < \frac{1}{2d} \frac{\Delta x}{\max|U|},\tag{4.2}$$

where d is the dimension of the problem (2 or 3). We first consider step 1 above. Without loss of generality, we consider step 1 (4.1) in one dimension with positive velocity u. Then one has

$$A_i^* = A^n(x - u\Delta t) = \frac{u\Delta t}{\Delta x} A_{i-1}^n + \left(1 - \frac{u\Delta t}{\Delta x}\right) A_i^n.$$
(4.3)

The stability constraint (4.2) insures that the coefficients of A in (4.3) are positive which in turn insures that A^* is positive definite symmetric since the linear combination of positive definite symmetric matrices, with positive coefficients, is also positive definite symmetric. In step 2 of Sect. 4 above, the stability constraint (4.2) insures that the matrix,

$$S = I + \Delta t [\nabla U^L]_{i,i}^n,$$

is nonsingular. Since *S* is nonsingular, and A^* is positive definite symmetric, this implies that the matrix $A^{**} = SA^*S^T$ is also a positive definite symmetric matrix. In step 3 of Sect. 4 above, we have the linear combination of two positive definite symmetric matrices, with positive coefficients, which insures that A^{n+1} is also positive definite symmetric. Finally, the extrapolation step (step 4) preserves the extrapolated values of A^{n+1} as positive definite symmetric since we are implementing a piecewise constant extrapolation algorithm. We note that for our choice of f(R) in (1.2), we have to treat the special case when trace $(A) \ge L^2$; in this case we assume a relaxation time of zero and therefore replace the configuration tensor *A* with the identity matrix *I*.

5 Matrix Solver

The equations resulting from the calculations of the viscous force terms or the pressure projection steps can be written generally as,

$$\alpha(x)p + \beta \nabla \cdot (A(x, y)\nabla p) = f(x, y) + \alpha(x)q + \beta \nabla \cdot V,$$
(5.1)

where A is a diagonal matrix. We discretize (5.1) as,

$$\alpha_{ij} p_{ij} \beta (DGp)_{ij} = f_{ij} + \alpha_{ij} q_{ij} + \beta (DV)_{ij}, \qquad (5.2)$$

where

$$(Gp)_{i+1/2,j} = (A^{11})_{i+1/2,j} \frac{p_{i+1,j} - p_{ij}}{\Delta x}, \qquad (Gp)_{i,j+1/2} = (A^{22})_{i,j+1/2} \frac{p_{i,j+1} - p_{ij}}{\Delta y},$$
$$(DV)_{i,j} = \frac{u_{i+1/2,j} - u_{i-1/2,j}}{\Delta x} + \frac{v_{i,j+1/2} - v_{i,j-1/2}}{\Delta y}.$$

We solve (5.2) with a combination of the multigrid method and the multigrid preconditioned conjugate gradient method (MGPCG) [21]. The adaptive composite solver will converge to any specified tolerance ε , as long as we compute the solution on each level (using MGPCG) to a tolerance of $\varepsilon \times 10^{-2}$ each time the outer multigrid solver visits a level. Our Multigrid+MGPCG algorithm for solving (5.2) follows the same "V-cycle" procedure as outlined by Briggs, et al. [22]:

Put (5.2) in residual correction form:

For $l = 0 \dots l_{\text{max}}$ $V^{l} = V^{l} - Gp_{\text{predict}}^{l}$ $q^{l} = q^{l} - p_{\text{predict}}^{l}$ $p^{l} = p_{\text{predict}}^{l}$ EndFor

Then, call recursive subroutine MV(l):

subroutine MV(*l*)

1. if $l < l_{max}$ then restrict V^{l+1} to level l2. $R^{l} = f^{l} + \alpha q^{l} + \beta D V^{l}$ 3. if $l < l_{max}$ then restrict R^{l+1} to level l4. $V_{save}^{l} = V^{l}, q_{save}^{l} = q^{l}, R_{save}^{l} = R^{l}$ 5. $p_{cor}^{l} = 0$ 6. Iterate until the residual on level l is less than the tolerance $\varepsilon \times 10^{-2}$.

Iterate using MGPCG,

$$\alpha p_{\text{cor}}^{l} + \beta DG p_{\text{cor}}^{l} = R_{\text{save}}^{l}$$
7.
$$R^{l} = R_{\text{save}}^{l} - \alpha p_{\text{cor}}^{l} - \beta DG p_{\text{cor}}^{l}, V^{l} = V^{l} - G p_{\text{cor}}^{1}, q^{l} = q^{l} - p_{\text{cor}}^{l}$$
8. if $l > 0$ then
$$\begin{cases} \text{call } MV(l-1) \\ \text{prolongate } p_{\text{cor}}^{l-1} \text{ to level } l, \ p_{\text{cor}}^{l} = p_{\text{cor}}^{l} + I_{l-1}^{l} p_{\text{cor}}^{l-1} \end{cases}$$

9. Iterate until the residual on level *l* is less than the tolerance $\varepsilon \times 10^{-2}$. Iterate using MGPCG,

 $\alpha p_{\text{cor}}^{l} + \beta DG p_{\text{cor}}^{l} = R_{\text{save}}^{l}$ 10. $V^{l} = V_{\text{save}}^{l} - G p_{\text{cor}}^{l}, q^{l} = q_{\text{save}}^{l} - p_{\text{cor}}^{l}, p^{l} = p^{l} + p_{\text{cor}}^{l}$

6 Overall Numerical Algorithm for Two-Phase Viscoelastic/Viscous Flow

Prior to each time step, we are given a liquid velocity, $U^{L,n}$, and a total velocity U^n . $U^{L,n}$ corresponds to U^n except on gas faces, where we replace the gas velocity in $U^{L,n}$ with the extrapolated liquid velocity. $U^{L,n}$ is then used to calculate the nonlinear advective terms in the liquid and also used to advance the free surface.

Prior to each time step, we are also given a level set function, φ^n and a volume-of-fluid function F^n . The level set function and the volume-of-fluid function are stored at cell centers. The velocity is stored at both cell centers and face-centers.

An outline of our numerical algorithm is as follows:

1. CLSVOF; interface advection [1]:

$$\varphi_{ij}^{n+1} = \varphi_{ij}^n - \Delta t [U^L \cdot \nabla \phi]_{ij}, \qquad F_{ij}^{n+1} = F_{ij}^n - \Delta t [U^L \cdot \nabla F]_{ij}.$$

2. Calculate (cell centered) advective force terms:

$$A_{ij}^{L} = [U^{L} \cdot \nabla U^{L}]_{ij}^{n}, \qquad A_{ij} = [U \cdot \nabla U]_{ij}^{n}.$$

The nonlinear advective terms are discretized using upwind, second order Van-Leer slop limiting [1, 23].

3. Update the configuration tensor (viscoelastic flows):

$$A_{i,j}^{n+1} = \frac{\Delta t \frac{f(R)}{\lambda} I + (SA^n (\vec{x} - \vec{u}\Delta t)S^T)_{i,j}}{1 + \Delta t \frac{f(R)}{\lambda}}, \quad S = I + \Delta t \nabla U.$$
(6.1)

Please see Sect. 4 for details that describe the discretization of (6.1).

4. Calculate (cell centered) semi-implicit viscous forces and explicit viscoelastic forces:

$$\begin{split} U_{i,j}^{n} &= \begin{cases} U_{i,j}^{L,n}, & \phi_{i,j} \geq 0, \\ U_{i,j}^{n}, & \phi_{i,j} < 0, \end{cases} \qquad A_{ij} = \begin{cases} A_{i,j}^{L}, & \phi_{i,j} \geq 0, \\ A_{i,j}, & \phi_{i,j} < 0, \end{cases} \\ U_{i,j}^{*} &= U_{i,j}^{n} + \Delta t \left(-A_{i,j} + L_{i,j}^{\text{viscoelastic}} + g \hat{z} \right), \\ \rho \frac{U^{**} - U^{*}}{\Delta t} &= L^{**,\text{uncoupled}} + L^{*,\text{uncoupled}}. \end{split}$$

The term $L^{**,\text{uncoupled}}$ represents the uncoupled viscous force terms which are handled implicitly, and the term $L^{*,\text{coupled}}$ represents the coupled viscous force terms which are handled explicitly.

5. Interpolate cell centered forces to face centered forces and calculate the face centered surface tension force:

$$V_{i+1/2,j} = U_{i+1/2,j}^n + \Delta t \left(-A_{i+1/2,j} + \frac{2}{\rho_{i+1,j} + \rho_{i,j}} (L_{i+1/2,j} + L_{i+1/2,j}^{\text{viscoelastic}}) - \left[\frac{\sigma \kappa \nabla H}{\rho} \right]_{i+1/2,j} + g\hat{z} \right).$$

The Surface tension term, $\frac{\sigma_{\kappa_{i+1/2,j}}(\nabla H)_{i+1/2,j}}{\rho_{i+1/2,j}}$, is discretized as

$$\sigma \kappa_{i+1/2,j} \frac{H(\phi_{i+1,j}) - H(\phi_{i,j})}{\rho_{i+1/2,j} \Delta x}, \text{ where } H(\phi) = \begin{cases} 1, & \phi \ge 0, \\ 0, & \phi < 0, \end{cases} \text{ and } \\ \rho_{i+1/2,j} = \rho_L \theta_{i+1/2,j} + \rho_G (1 - \theta_{i+1/2,j}). \end{cases}$$

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Table 1Comparison ofcomputational results with		Previous	New
analytical solution for $Re \ll 1$	Computational <i>Re</i> number	0.0524	0.0524
	Analytical solution by Hadamard-Rybczynski	0.0534	0.0534

The discretization of the height fraction $\theta_{i+1/2,j}$ is described by (3.2). The curvature $\kappa_{i+1/2,j}$ is computed with second-order accuracy directly from the volume fractions as described in [24] or [1].

6. Implicit pressure projection step:

$$\nabla \cdot \frac{\nabla p}{\rho} = \nabla \cdot V, \qquad U_{i+1/2,j}^{n+1} = V_{i+1/2,j} - \left[\frac{\nabla p}{\rho}\right]_{i+1/2,j}.$$

We solve the resulting linear system using a composite solver for an adaptive hierarchy of grids [25] (see also Sect. 5).

- 7. Liquid velocity extrapolation; assign $U_{i+1/2,j}^{L,n+1} = U_{i+1/2,j}^{n+1}$ and then extrapolate $U_{i+1/2,j}^{L,n+1}$ into the gas region.
- 8. Interpolate face centered velocity to cell centered velocity:

$$U_{i,j}^{L,n+1} = \frac{1}{2} \Big(U_{i+1/2,j}^{L,n+1} + U_{i-1/2,j}^{L,n+1} \Big), \qquad U_{i,j}^{n+1} = \frac{1}{2} \Big(U_{i+1/2,j}^{n+1} + U_{i-1/2,j}^{n+1} \Big).$$

7 Results

7.1 Spherical Bubble or Drop with Dominant Viscous Effects and Large Viscosity Jump: Comparison with Analytical and Experimental Results

For a spherical bubble, drop and particle freely moving relative to a fluid of infinite extent with a steady velocity with $Re \ll 1.0$, the analytical solution derived by Hadamard-Rybczynsk [26, 27] is well-known as follows:

$$Re = \frac{Eo^{1.5}}{6M^{0.5}} \left(\frac{1+\kappa}{2+3\kappa}\right).$$
 (7.1)

Here, *Eo* is the Eotvos number, *M* denotes the Morton number and $\kappa (= \mu_D/\mu_C)$ is the viscosity ratio. Equation (7.1) reduces to $Re = \frac{Eo^{1.5}}{12M^{0.5}}$ as $\kappa \to 0$ and becomes $\frac{Eo^{1.5}}{18M^{0.5}}$ for $\kappa \to \infty$. We show results for a spherical bubble with low Reynolds number. The computational condition is $\log M = 2.85$, Eo = 8.67 which is an experimental condition performed by Bhaga and Weber [28]. Although they examined bubble rise dynamics, in this computation, we changed to a density ratio $\lambda (= \rho_D/\rho_C) = 0.95$ and $\kappa = 100$ to clearly specify a problem with a large viscosity jump. The numerical simulations were performed on a 3-d axisymmetric computational domain with a domain length of R = 4D (*r*-direction) and a domain height of Z = 6D (*z*-direction). In the computation, one mesh size on the finest level grid was set to $\Delta x = 0.0104$ in terms of the dimensionless bubble/drop diameter (D = 1.0). Comparison of computational results with analytical solution is indicated in Table 1. The error between our computational results and analytical solution is only about 2%. We reiterate that the fluid properties for this problem exhibit very large viscosity.



We then recomputed the previous example ($\lambda = 0.95$, $\kappa = 100$), except with logM = 2.93 and Eo = 116; this test was also performed by Bhaga and Weber [28]. Figure 2 shows a comparison of results using our previous method from [3] and the new methodology described in this paper. Computational conditions are the same as the case for low *Re* number. In Fig. 2, the left side is the previous method and the right side corresponds to the new method.

7.2 Drop Formation in a Miscible Viscous Liquid

Figure 3 corresponds to the drop formation process in a miscible viscous liquid. Bright [29] injected n-decane into water from a submerged nozzle with $d = 1.6 \times 10^{-3}$ m, and he investigated the drop formation process in the system ($\lambda = 0.7327$ and $\kappa = 0.90$). Richards et al. [30] numerically calculated the Bright's experimental result on the drop formation process using the SOLA-VOF method [31] with the continuum surface force (CFS) model [32]. In their computation, Richards et al. [30] used the value of the interfacial tension experimentally estimated by Johnson and Dettre [33] instead of the interfacial tension measured by Bright [29]. Our computational conditions for the physical properties follow the paper of Richards et al. [30]. The computations were carried out in a 3-d axisymmetric computational domain with a domain length of R = 16d (*r*-direction) and a domain height of Z = 32d (*z*-direction). Also, one mesh size on the finest level grid was $\Delta x = 0.0125$ in terms of the dimensionless diameter of nozzle (= 1.0). In Fig. 3, the left side is the previous method [3] and the right side corresponds to the new method. We have confirmed that the appearance of the drop formation reproduced numerically agrees with the experimental result.

7.3 Bubble Formation; Injection of High Speed Gas into Liquid

In test problems with fluids that have a large viscosity, such as the test problem described in Sect. 7.1 above, a fully implicit treatment of the coupled viscous terms would lead to

Fig. 3 Drop formation in a viscous liquid. *Left*: old method, *right*: new method



Table 2 Grid resolution study for bubble formation problem. Relative errors are displayed. Inflow velocity is 0.9 m/s; density ratio = 1015, viscosity ratio = 70000

Eff. Grid Resolution	64×128	128×256	256 × 512
Interface Error		0.74	0.45
Gas Velocity Error		1.44	0.65

a matrix system that could not be solved via a standard matrix iterative approach. On the other hand we have found that, for the bubble formation problem, the explicit treatment of the viscous coupling terms, as described by [3], leads to instabilities. Here, we perform a grid refinement study using our new approach for a problem that we had previously failed to compute using the approach described in [3]. The physical properties of the system, together with the inflow velocity condition, are given by the following dimensionless quantities,

$$Re_{G} = \frac{\rho_{G}Vd}{\mu_{G}} = 99.2, \qquad We_{G} = \frac{\rho_{G}V^{2}d}{\sigma} = 0.024, \qquad Fr = \frac{V^{2}}{gd} = 46.6,$$
$$\frac{\rho_{L}}{\rho_{G}} = 1015, \qquad \frac{\mu_{L}}{\mu_{G}} = 70000.$$

The relative errors in Table 1 indicate that we have first order convergence for this problem using our new treatment for the viscous coupling terms.



Fig. 5 Computational setup; dynamic adaptive gridding used around the viscoelastic bubble

7.3.1 Viscoelastic Bubbles

We consider a gas-liquid system with the physical properties defined by the Morton number $M (=g\eta_S^4(\rho_S - \rho_G)/\rho_S^2\sigma^3) = 78$, the density ratio $\rho_S/\rho_G = 845$ and the viscosity ratio $\eta_S/\eta_G = 55000$. We shall investigate the effect of FENE–CR model parameters given these physical properties. The computational domain for the 2d-axisymmetric computations is shown schematically in Fig. 5. In terms of the dimensionless diameter of bubble d = 1.0, the computational domain has an *r*-directional dimensionless length R = 3.0 and a *z*-directional dimensionless height H = 7.5. In our computations, as initial conditions, a spherical bubble was artificially imposed at H = 4.5. The physical boundary conditions were as follows: inflow at the top wall, outflow boundaries at the bottom and right walls, and reflecting boundary conditions at the axisymmetric center (r = 0).



Fig. 7 Left: transient dynamics of bubble for L = 20, c = 0.1, De = 5.0. Right: distribution of tr(A) for Eo = 19.6, L = 10, c = 0.1, De = 5.0

Figure 5 is a schematic drawing of the AMR system. The left side of Fig. 5 shows an example of the grid structure and the right side of Fig. 5 shows a computational example of a 2d-axisymmetric bubble rising. The mesh hierarchy is composed of different levels of refinement ranging from coarsest ($\ell = 0$, we label as "Level 0".) to finest ($\ell = \ell_{max}$), and the refinement ratio between levels is 2; i.e. $\Delta x^{\ell+1} = 0.5\Delta x^{\ell}$. In Fig. 5, five levels (Level 0–4) are indicated. In the computations, one mesh size on the finest adaptive level Δx^{fine} was about 0.0078 in terms of d = 1.0. The value $\Delta x^{fine} = 0.0078$ was adopted as a result of a mesh refinement study for the stiffest flow condition. This means there are (effectively) about 85 cells per bubble diameter.

Figure 6 shows bubble shapes depending on the Deborah number $De (= V\lambda/D, V$: characteristic velocity (bubble rise speed), D: bubble diameter). The upper figures are obtained under the condition of the Eötvös number $Eo (= g(\rho_S - \rho_G)D^2/\sigma) = 19.6, L = 5.0, c = 0.1$ and the lower figures use Eo = 78.3, L = 10.0, c = 0.1. As a basis of comparison, the results for the corresponding Newtonian bubble are indicated together with the viscoelastic results. For the largest De number, bubbles have the well-known sharp cusp-like trailing edge. Especially, regardless of the largely dimpled Newtonian bubble for Eo = 78.3, it is surprising that the bubble for L = 10.0, c = 0.1, De = 5.0 has a very long sharp cusp. This result indicates that our computational method can capture largely deformed shapes having large curvature. It is verified that viscoelastic effects (cusp shape) emerge notably as the De number becomes larger. Noh et al. [34] gave good physical explanations about the effect of the parameters of the FENE-CR model, and our results support their observations. Figure 7 shows transient dynamics of bubble motion for the condition of L = 20, c = 0.1, De = 5.0. In this case, the bubble trailing edge is pulled out due to the strong viscoelastic effect (viscoelastic stress). Consequently, the pulling motion leads to the local breakup of the bubble at the trailing edge. As an example of the tr(A) profile, the distribution of tr(A) for Eo = 19.6, L = 10, c = 0.1, De = 5.0 is shown as well in Fig. 7. tr(A) is a measure of the degree of polymer extension. In Fig. 7, a contour plot of tr(A) is shown in which 40 contours are plotted ranging from the maximum trace (tr(A) = 95.0) to the minimum trace (tr(A) = 3.0). The outer contour corresponds to the minimum value. As is clear in Fig. 7, the value of tr(A) is relatively large near the interface and that the local maximums of tr(A) are near the top and bottom surfaces. Due to these viscoelastic stresses, it can be recognized that the trailing cusp at the bottom is formed and the vertical bubble length increases.

8 Conclusions

We have tested two improvements for discretizing the viscosity force terms, and viscoelastic force terms as they appear in the Navier-Stokes equations for two-phase flows. These improvements preserve the property of our sharp interface approach in that solutions of the two-phase problem approach solutions of the corresponding one-phase problem in the limit of zero gas density and zero gas viscosity. These two improvements enable us to robustly calculate viscous and viscoelastic two-phase flows over a very large range of physical properties of the flow. For example, in Sect. 7.1 we have validated our approach for problems where $Re \ll 1$. In Sect. 7.3, we validated our new approach for a bubble formation problem with a relatively large Reynolds number: $Re_G = 99.2$. The method described by [3] becomes unstable for the bubble formation problem. As illustrated by Fig. 3 and Fig. 7, our coupled level-set and volume-of-fluid approach for representing and updating the interface can accurately handle sharp corners, and changes in interface topology.

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