1 Solutions Test 1, MAP2302

1. 

\[ ty' + 5y = t^3 \quad y(1) = 1 \]
\[ y' + (5/t)y = t^2 \]
\[ \mu = e^{\int (5/t)dt} = e^{5\ln t} = t^5 \]
\[ t^5y = \int t^5 \text{ } dt = (1/8)t^8 + C \]
\[ y = (1/8)t^3 + C/t^5 \]
\[ 1 = (1/8) + C \quad C = 7/8 \]
\[ y = (1/8)t^3 + (7/8)1/t^5 \]

Solution valid for \( 0 < t < \infty \).

\[ \lim_{t \to 0^+} (1/8)t^3 + (7/8)1/t^5 = \infty \]
\[ \lim_{t \to \infty} (1/8)t^3 + (7/8)1/t^5 = \infty \]

2. 

\[ y' = -cy^2 + (a - m)y - h \equiv f(y) \]
equilibrium points when \( f(y) = 0 \):

\[ -cy^2 + (a - m)y - h = 0 \]
\[ y = \frac{(m - a) \pm \sqrt{(a - m)^2 - 4(-c)(-h)}}{2(-c)} \]
\[ y = \frac{(a - m) \pm \sqrt{(a - m)^2 - 4ac}}{2c} \]

For there to be two distinct roots,

\[ (a - m)^2 - 4ac > 0 \]
\[ 4ch < (a - m)^2 \]
\[ h < (a - m)^2/(4c) \]
Let \( y_1 = \frac{(a-m)-\sqrt{(a-m)^2-4ch}}{2c} \) and \( y_2 = \frac{(a-m)+\sqrt{(a-m)^2-4ch}}{2c} \). For \( y < y_1 \), \( f(y) < 0 \) (this is found by checking the sign of \( f(y) \) as \( y \to -\infty \)). For \( y > y_2 \), \( f(y) < 0 \) (this is found by checking the sign of \( f(y) \) as \( y \to \infty \)). For \( y_1 < y < y_2 \), \( f(y) > 0 \). This can be found by checking the sign of \( f(y) \) where \( y = (y_1 + y_2)/2 \):

\[
\frac{y_1 + y_2}{2} = \frac{a-m}{2c}
\]

\[-c\left(\frac{a-m}{2c}\right)^2 + (a-m)\frac{a-m}{2c} - h = \frac{(a-m)^2}{4c} - h > 0
\]

Since \( y' < 0 \) for \( y < y_1 \) and \( y' > 0 \) for \( y > y_1 \), \( y_1 \) is unstable. Since \( y' > 0 \) for \( y < y_2 \) and \( y' < 0 \) for \( y > y_2 \), \( y_2 \) is stable.

3.

\[
dy/dx = \frac{3x^2 + 2x + 4}{2(y-2)} \quad y(0) = 0
\]

\[
(2y - 4)dy = (3x^2 + 2x + 4)dx
\]

\[
y^2 - 4y = x^3 + x^2 + 4x + C
\]

\[
0 - 0 = 0 + 0 + 0 + C \quad C = 0
\]

\[
y^2 - 4y = x^3 + x^2 + 4x
\]

4.

\[
t^2y' + 2ty - y^3 = 0 \quad y(1) = 1
\]

\[
t^2dy + (2ty - y^3)dt = 0
\]

\[
\partial M/\partial y = 2t - 3y^2
\]

\[
\partial N/\partial t = 2t
\]

\[
\partial M/\partial y \neq \partial N/\partial t
\]

\[
v = 1/y^2
\]

\[
v' = (y^{-2})' = -2y^{-3}y'
\]

\[
y' = v'y^3/(-2)
\]

\[
t^2(v'y^3/(-2)) + 2ty - y^3 = 0
\]

\[
t^2v'/(-2) + 2ty^2 - 1 = 0
\]

\[
t^2v'/(-2) + 2tv - 1 = 0
\]
\[ t^2 v' - 4tv + 2 = 0 \]
\[ v' - \frac{4}{t}v + \frac{2}{t^2} = 0 \]
\[ v' - \frac{4}{t}v = \frac{-2}{t^2} \]
\[ \mu = e^{\int -\frac{4}{t}dt} = e^{-4\ln t} = \frac{1}{t^4} \]

\[ \frac{v}{t^4} = \int \frac{-2}{t^2} (1/t^4) dt \]
\[ \frac{v}{t^4} = \int -2/t^6 dt \]
\[ \frac{v}{t^4} = (2/5)t^{-5} + C \]
\[ v = (2/5)(1/t) + Ct^4 \]
\[ 1/y^2 = (2/5)(1/t) + Ct^4 \]
\[ 1 = (2/5) + C \quad C = 3/5 \]
\[ 1/y^2 = (2/5)(1/t) + (3/5)t^4 \]