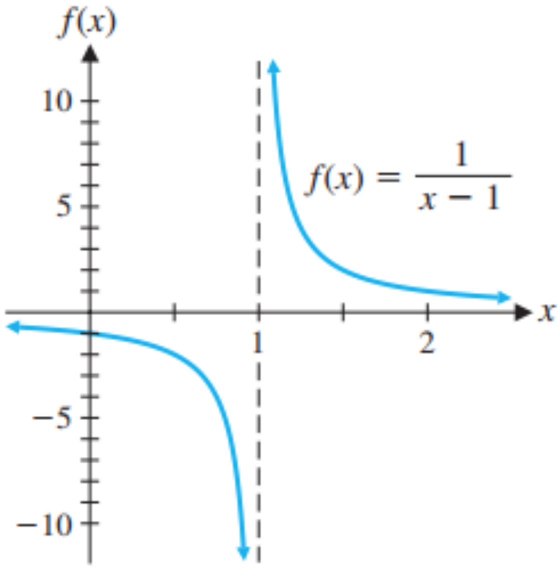
2.2. Infinite Limits and Limits at Infinity

Example 2.2.1. *Find the limit* $\lim_{x \to 1} \dfrac{1}{x-1}$



Definition 2.2.1. *The vertical dotted line* $x = 1$ *in the above example is a* **vertical asymptote**.

Infinite Limits and Vertical Asymptotes

Definition 2.2.2. *If* $f(x)$ *fails to exist as* x *approaches* a *from the left because the values of* $f(x)$ *are becoming very large positive numbers (or very large negative numbers) then we say*

$$\lim_{x \to a^-} f(x) = \infty \text{ or } (-\infty)$$

If the above behavior happens when x *approaches* a *from the right then we say*

$$\lim_{x \to a^+} f(x) = \infty \text{ or } (-\infty)$$

If both one-sided limits exhibit same behavior then we say

$$\lim_{x \to a} f(x) = \infty \text{ or } (-\infty)$$

Moreover, in these cases the graph $y = f(x)$ *has a vertical asymptote at* $x = a$.

Steps for Determining Vertical Asymptotes Given Equations

(1) Simplify the equation completely by factoring
(2) Determine where the denominator is 0
(3) (For limit problems) For each value found in last step, plug in numbers very close to the left and right of each value to determine sign (positive or negative). This tells you if left-/right- handed limits are positive or negative infinity.

Example 2.2.2. *Find the limits* $\lim_{x\to 0^+} \frac{1}{x}$ *and* $\lim_{x\to 0^-} \frac{1}{x}$

Example 2.2.3. $\lim_{x\to 4^-} \frac{3}{x-4}$

Example 2.2.4. $\lim_{x\to 4^+} \frac{3}{x-4}$

Example 2.2.5. $\lim_{x\to -1^-} \frac{-5}{x+1}$

Example 2.2.6. $\lim_{x\to -1^+} \frac{-5}{x+1}$

Example 2.2.7. *Find the vertical asymptotes of* $f(x) = \dfrac{-5}{x+1}$

Example 2.2.8. $\displaystyle\lim_{x\to 2^+} \frac{x^2-2x-3}{x^2-x-2}$

Example 2.2.9. $\displaystyle\lim_{x\to -1^+} \frac{x^2-2x-3}{x^2-x-2}$

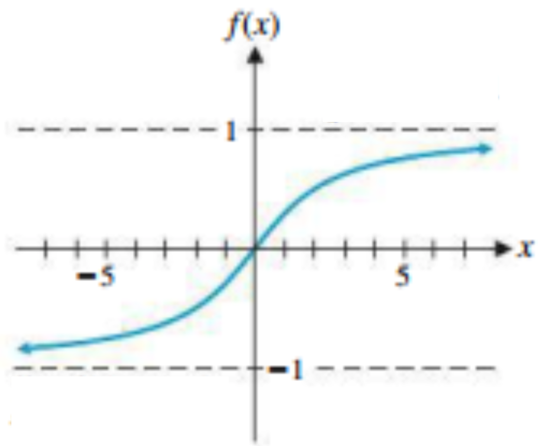
Example 2.2.10. *Find the vertical asymptotes of* $f(x) = \dfrac{x^2-2x-3}{x^2-x-2}$

Limits at Infinity

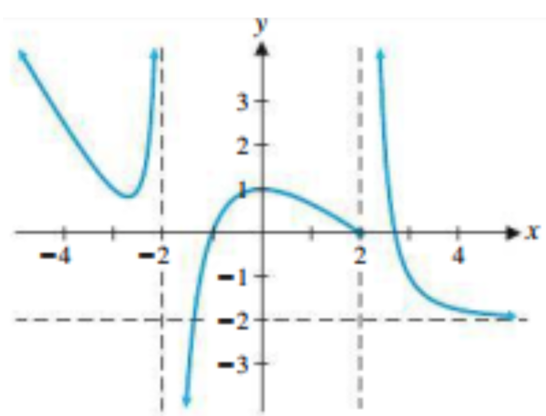
Definition 2.2.3. *We say the limit as x approaches infinity is L, written $\lim_{x\to\infty} f(x) = L$, if for some x large enough the graph of $y = f(x)$ moves closer and closer to the line $y = L$ as one moves to the right. Moreover, in this case the graph $y = f(x)$ has a horizontal asymptote $y = L$. (L must be a real number!)*

Definition 2.2.4. *We say the limit as x approaches negative infinity is L, written $\lim_{x\to-\infty} f(x) = L$, if for some x far enough to the left the graph of $y = f(x)$ moves closer and closer to the line $y = L$ as one moves further to the left. Moreover, in this case the graph $y = f(x)$ has a horizontal asymptote $y = L$. (L must be a real number!)*

Example 2.2.11. *Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ for the following function.*



Example 2.2.12. *Find the infinite limits, $\lim_{x\to\infty} f(x)$, $\lim_{x\to-\infty} f(x)$, the vertical asymptotes, and the horizontal asymptotes for the following function.*



Steps for Determining Limits at Infinity Given Equations

(1) If $f(x)$ is a polynomial, then $\lim_{x\to\pm\infty} f(x) = \pm\infty$. The sign is determined by the leading term.

(2) If $f(x)$ is a rational function, then
  (a) $\lim_{x\to\pm\infty} f(x) = \pm\infty$ if degree of numerator is higher than degree of denominator (Use the leading term in the numerator AND denominator to determine the sign)

  (b) $\lim_{x\to\pm\infty} f(x) = 0$ if degree of denominator is higher than degree of numerator

  (c) $\lim_{x\to\pm\infty} f(x) = a/b$ if degree of numerator is equal to degree of denominator and a is the leading coefficient of numerator and b is leading coefficient of denominator.

Example 2.2.13. *Find the limits* $\lim\limits_{x\to\infty} 4x^5$ *and* $\lim\limits_{x\to-\infty} 4x^5$

Example 2.2.14. *Find the limits* $\lim\limits_{x\to\infty} -4x^5$ *and* $\lim\limits_{x\to-\infty} -4x^5$

Example 2.2.15. *Find the limits* $\lim\limits_{x\to\infty} -4x^8$ *and* $\lim\limits_{x\to-\infty} -4x^8$

Example 2.2.16. *Find the limits* $\lim\limits_{x\to\infty} (-2x^3 + 4x^2 - x + 5)$ *and* $\lim\limits_{x\to-\infty} (-2x^3 + 4x^2 - x + 5)$

Example 2.2.17. *Find the limits* $\lim\limits_{x\to\infty} (-2x^3 + 4x^5 - x + 5)$ *and* $\lim\limits_{x\to-\infty} (-2x^3 + 4x^5 - x + 5)$

Example 2.2.18. *Find the limit* $\lim_{x \to -\infty} \dfrac{-2x^2 + 2}{5x^3 + x^2 - 1}$

Example 2.2.19. *Find the horizontal asymptote(s) of* $f(x) = \dfrac{-2x^2 + 2}{5x^3 + x^2 - 1}$

Example 2.2.20. *Find the limit* $\lim_{x \to \infty} \dfrac{-2x^3 + x^2 - 1}{3x^2 + 2}$

Example 2.2.21. *Find the horizontal asymptote(s) of* $f(x) = \dfrac{-2x^3 + x^2 - 1}{3x^2 + 2}$

Example 2.2.22. *Find the limit* $\lim_{x \to \infty} \dfrac{3x^5 + 2x^3 - 1}{6x^5 + 7}$

Example 2.2.23. *Find the horizontal asymptote(s) of* $f(x) = \dfrac{3x^5 + 2x^3 - 1}{6x^5 + 7}$

Example 2.2.24. *Find all horizontal and vertical asymptotes for* $f(x) = \dfrac{x}{x^2 - 4}$

Example 2.2.25. *Find all horizontal and vertical asymptotes for* $f(x) = \dfrac{2x^2 + 7x + 12}{2x^2 + 5x - 12}$