Stolarsky Principle: generalizations, extensions, and applications.

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Point Distributions Webinar

September 16, 2020

Spherical cap discrepancy

For $x \in \mathbb{S}^d$, $t \in [-1,1]$ define spherical caps:

$$C(x,t) = \{y \in \mathbb{S}^d : \langle x, y \rangle \geq t\}.$$

For a finite set $Z = \{z_1, z_2, ..., z_N\} \subset \mathbb{S}^d$ define

$$D_{cap}(Z) = \sup_{x \in \mathbb{S}^d, t \in [-1,1]} \left| \frac{\#\big(Z \cap C(x,t)\big)}{N} - \sigma\big(C(x,t)\big) \right|.$$

Theorem (Beck, '84)

There exists constants c_d, $C_d > 0$ such that

$$c_d N^{-\frac{1}{2}-\frac{1}{2d}} \leq \inf_{\#Z=N} D_{cap}(Z) \leq C_d N^{-\frac{1}{2}-\frac{1}{2d}} \sqrt{\log N}.$$

Spherical caps: L^2 discrepancy

Define the spherical cap L^2 discrepancy

$$D_{cap,L^2}(Z) = \left(\int_{\mathbb{S}^d} \int_{-1}^{1} \left| \frac{\#\big(Z \cap C(x,t)\big)}{N} - \sigma\big(C(x,t)\big) \right|^2 dt\, d\sigma(x) \right)^{\frac{1}{2}}.$$

Theorem (Beck, '84)

There exists constants c_d, $C_d > 0$ such that

$$c_d N^{-\frac{1}{2}-\frac{1}{2d}} \leq \inf_{\#Z=N} D_{cap,L^2}(Z) \leq C_d N^{-\frac{1}{2}-\frac{1}{2d}}.$$

Discrepancy and energy: Stolarsky Principle

Theorem (Stolarsky invariance principle)

For any finite set $Z = \{z_1, ..., z_N\} \subset \mathbb{S}^d$

$$\frac{1}{N^2}\sum_{i,j=1}^{N} \|z_i - z_j\| \;+\; c_d\Big[D_{L^2,cap}(Z)\Big]^2 = \text{const}$$
$$= \int\limits_{\mathbb{S}^d}\int\limits_{\mathbb{S}^d} \|x-y\|\, d\sigma(x) d\sigma(y).$$

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$$c_d \Big[D_{L^2,cap}(Z) \Big]^2 =$$

$$= \int\limits_{\mathbb{S}^d} \int\limits_{\mathbb{S}^d} \|x - y\| \, d\sigma(x) d\sigma(y) \; - \; \frac{1}{N^2} \sum_{i,j=1}^{N} \|z_i - z_j\|.$$

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- Stolarsky '73, Brauchart, Dick '12, DB, Dai, Matzke '18

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- Stolarsky '73, Brauchart, Dick '12, DB, Dai, Matzke '18
- $C_d = c_d^{-1} = \dfrac{1}{2} \displaystyle\int_{\mathbb{S}^d} |p \cdot z| d\sigma(z) = \dfrac{1}{d} \dfrac{\omega_{d-1}}{\omega_d} \approx \dfrac{1}{\sqrt{2\pi d}}.$

Discrepancy and energy: Stolarsky Principle

Theorem (Stolarsky invariance principle)

For any finite set $Z = \{z_1, ..., z_N\} \subset \mathbb{S}^d$

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$$= \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} \|x - y\| \, d\sigma(x) d\sigma(y) \; - \; \frac{1}{N^2} \sum_{i,j=1}^{N} \|z_i - z_j\|.$$

- Stolarsky '73, Brauchart, Dick '12, DB, Dai, Matzke '18
- $C_d = c_d^{-1} = \dfrac{1}{2} \displaystyle\int_{\mathbb{S}^d} |p \cdot z| d\sigma(z) = \dfrac{1}{d} \dfrac{\omega_{d-1}}{\omega_d} \approx \dfrac{1}{\sqrt{2\pi d}}$.
- Easy corollaries:
  - i.i.d. random points: $\mathbb{E} D^2_{L^2,cap}(Z) \lesssim N^{-1}$
  - jittered sampling: $\mathbb{E} D^2_{L^2,cap}(Z) \lesssim N^{-1-\frac{1}{d}}$

Proof:

Discrete energy

Let $Z = \{z_1, ..., z_N\} \subset \mathbb{S}^d$ and let $F : [-1, 1] \to \mathbb{R}$.
Discrete energy:

$$E_F(Z) = \frac{1}{N^2} \sum_{i,j=1}^{N} F(z_i \cdot z_j)$$

- $F(x \cdot y) = -\|x - y\|$:
  sum of distances (Fejes-Tóth Problem): on $\mathbb{S}^2$ minimizers known for $N = 2, 3, 4, 5, 6$ and $N = 12$.

Questions:

- What are the minimizing configurations?
- Almost minimizers?
- Lower bounds?

Energy integral

Let μ be a Borel probability measure on $\mathbb{S}^d$.
Energy integral

$$I_F(\mu) = \int\limits_{\mathbb{S}^d} \int\limits_{\mathbb{S}^d} F(x \cdot y)\, d\mu(x) d\mu(y).$$

i.e. $E_F(Z) = I_F\left(\frac{1}{N} \sum \delta_{z_i}\right)$

Questions:

- What are the minimizers?
- Is σ a minimizer?
- Is it unique?

Other versions, extensions, generalizations

$$\Big(\ L^2 \text{ discrepancy } \Big)^2 \ = \ \text{distance integral} \ - \ \text{sum of distances},$$

$$\Big(\ L^2 \text{ discrepancy } \Big)^2 \ = \ \text{discrete energy} \ - \ \text{energy integral}.$$

$$\Big(\ L^2 \text{ discrepancy of } \mu \ \Big)^2 \ = \ \text{energy of } \mu - \ \text{optimal energy}$$

Original generalized version (Stolarsky)

For $x,\ y \in \mathbb{S}^d$ define the distance $\rho(x,y)$ as

$$\rho(x,y) = \int\limits_{\mathbb{S}^d} \int\limits_{\min\{x\cdot z, y\cdot z\}}^{\max\{x\cdot z, y\cdot z\}} g(t)\, dt\, d\sigma(z).$$

Then for any set $Z = \{z_1, ..., z_N\} \subset \mathbb{S}^d$ we have

$$\frac{1}{N^2} \sum_{i,j=1}^{N} \rho(z_i, z_j) + 2D^2_{L^2, cap,\, g}(Z) = \int\limits_{\mathbb{S}^d} \int\limits_{\mathbb{S}^d} \rho(x,y)\, d\sigma(x)\, d\sigma(y),$$

where $D_{L^2,\mathrm{cap},\, g}(Z)$ is the g-weighted L^2 spherical cap discrepancy of Z:

$$D^2_{L^2,\mathrm{cap},\, g}(Z) = \int\limits_{-1}^{1} g(t) \int\limits_{\mathbb{S}^d} \left| \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}_{C(z,t)}(z_i) - \sigma\big(C(p,t)\big) \right|^2 d\sigma(z)\, dt.$$

Original generalized version (Stolarsky)

- $g(t) = 1$:
  the standard spherical cap L^2-discrepancy, i.e.
  $D_{L^2,\text{cap},\mathbf{1}} = D_{L^2,\text{cap}}$,
  $\rho(x,y) = 2C_d\|x-y\|$ is a multiple of the Euclidean distance.
- The metric

$$d_1(x,y) = \int_{\mathbb{S}^d} \big|d(x,z) - d(y,z)\big| d\sigma(z),$$

where $d(\cdot,\cdot)$ is the geodesic distance, also has this form with $g(t) = (1-t^2)^{-1/2}$.

Further generalization: Brauchart

Case $n = 2$:
Let

$$\rho(x,y) = \int_{\mathbb{S}^d} \int_{\min\{x\cdot z, y\cdot z\}}^{\max\{x\cdot z, y\cdot z\}} \int_{\min\{x\cdot z, y\cdot z\}}^{\max\{x\cdot z, y\cdot z\}} g(t,t')\, dt\, dt'\, d\sigma(z).$$

Then

$$\frac{1}{N^2}\sum_{i,j} \rho(z_i, z_j) + 2\int_{-1}^{1}\int_{-1}^{1} g(t,t') D(Z;t,t')\, dt\, dt' = \int_{\mathbb{S}^d}\int_{\mathbb{S}^d} \rho(x,y) d\sigma\, d\sigma,$$

where

$$D(Z;t,t') = \int_{\mathbb{S}^d} D(Z, C(z,t)) D(Z, C(z,t')) d\sigma(z).$$

Application: even powers of Euclidean distances

Taking $g(u) = m(2m-1)u^{2(m-1)}$

$$\frac{1}{N^2}\sum_{i,j}\|z_i - z_j\|^{2m} \; + \; c_{d,m}\int_{-1}^{1}\int_{-1}^{1} g(|t-t'|)D(Z;t,t')\,dt\,dt'$$

$$= \int_{\mathbb{S}^d}\int_{\mathbb{S}^d}\|x-y\|^{2m}d\sigma(x)d\sigma(y)$$

Euclidean distance energy integrals

Let μ be a Borel probability measure on the sphere $\mathbb{S}^d$. For $\lambda > 0$ define the energy integral

$$I_\lambda = \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} \|x - y\|^\lambda d\mu(x) d\mu(y)$$

Maximizers (Bjorck '56):

- $0 < \lambda < 2$: unique maximizer is surface measure,
- $\lambda = 2$: any measure with center of mass at 0,
- $\lambda > 2$: mass $\frac{1}{2}$ at two opposite poles.

“Higher smoothness” (Brauchart and Dick, 2013)

$$K_\beta(x,y) = \int_{-1}^{1} \int_{\mathbb{S}^d} (x \cdot z - t)_+^{\beta-1} \, (y \cdot z - t)_+^{\beta-1} \; d\sigma(z) \, dt$$

Notice $K_1(x,y) = 1 - C_d \|x - y\|$

$$D^2_{L^2,\beta}(Z) := \int_{-1}^{1} \int_{\mathbb{S}^d} \left| \frac{1}{N} \sum_{j=1}^{N} (z_j \cdot z - t)_+^{\beta-1} - \int_{\mathbb{S}^d} (x \cdot z - t)_+^{\beta-1} d\sigma(x) \right|^2 dz dt$$

$$= \frac{1}{N^2} \sum_{i,j} K_\beta(z_i, z_j) \; - \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} K_\beta(x,y) d\sigma(x) d\sigma(y).$$

Application: odd powers of Euclidean distances

- Notice $K_1(x,y) = 1 - C_d\|x-y\|$
- If $\beta = M$ is a positive integer

$$K_M(x,y) = Q_{M-1}(x,y) \, + \, (-1)^M c_{d,M}\|x-y\|^{2M-1},$$

where Q_{M-1} is a polynomial of degree $M-1$.
- For any $Z = \{z_1, ..., z_N\} \subset \mathbb{S}^d$

$$D_{L^2,\beta}(Z) \geq c_{\beta,d} N^{-\frac{1}{2}-\frac{\beta-1/2}{d}}.$$

- For any spherical t-design Z:

$$D_{L^2,\beta}(Z) \leq C_{\beta,d} \left(t^d\right)^{-\frac{1}{2}-\frac{\beta-1/2}{d}}.$$

L^1 invariance principle (Skriganov)

Let M be a distance-invariant metric space of diameter π with a fixed probability measure μ.
Consider the "symmetric difference" metric

$$\theta^{\Delta}(\eta, x, y) = \int_0^{\pi} \theta_r^{\Delta}(x, y) \eta(r) dr,$$

where

$$\theta_r^{\Delta}(x, y) = \frac{1}{2} \mu\Big(B_r(x) \Delta B_r(y) \Big)$$

Once can also write

$$\theta_r^{\Delta}(x, y) = \frac{1}{2} \int_M \Big| \mathbf{1}_{B_r(x)}(z) - \mathbf{1}_{B_r(y)}(z) \Big| \, d\mu(z).$$

L^1 invariance principle (Skriganov)

L^2-discrepancy of an N-point set $Z \subset M$:

$$D_\eta^2(Z) := \int\limits_0^\pi \int\limits_M \Big(\frac{1}{N} \cdot \#Z \cap B_r(z) - \mu(B_r(y))\Big)^2 d\mu(z) \;\; \eta(r)\, dr.$$

L_1 invariance principle

$$D_\eta^2(Z) + \frac{1}{N^2} \sum_{x,y,\in Z} \theta^\Delta(\eta, x, y) = \int\limits_M \int\limits_M \theta^\Delta(\eta, x, y) d\mu(x) d\mu(y)$$

L^2 invariance principle (Skriganov)

Let Q be a connected compact two-point homogeneous space, i.e. $\mathbb{S}^d$, $\mathbb{RP}^n$, $\mathbb{CP}^n$, $\mathbb{HP}^n$, or $\mathbb{OP}^2$, with geodesic distance θ, normalized so that $\operatorname{diam}(Q) = \pi$. Define the *chordal distance*:

$$\tau(x, y) = \sin \frac{\theta(x, y)}{2}.$$

For $Q = \mathbb{S}^d$, $\tau(x, y) = \frac{1}{2}\|x - y\|$. For $Q = \mathbb{FP}^n$,

$$\tau(x, y) = \frac{1}{\sqrt{2}}\|\Pi(x) - \Pi(y)\|_F.$$

L^2 invariance principle (Skriganov)

- Let $\eta^\natural(r) = \sin r$. Then
$$\tau(x,y) = \gamma(Q)\theta^\Delta(\eta^\natural, x, y).$$
- Therefore, for any N-point set $Z \subset Q$:
$$\gamma(Q)\, D^2_{\eta^\natural}(Z) + \frac{1}{N^2}\sum_{x,y,\in Z} \tau(x,y) = \int_M\int_M \tau(x,y)d\mu(x)d\mu(y)$$
- If $Q = \mathbb{S}^d$, then $D_{\eta^\natural}(Z) = D_{L^2,cap}(Z)$.

Positive definite functions on the sphere

Lemma

For a function $F \in C[-1,1]$ the following are equivalent:

- i F *is positive definite on* $\mathbb{S}^d$.
- ii *Gegenbauer coefficients of* F *are non-negative, i.e.*

$$\widehat{F}(n,(d-1)/2) \geq 0 \text{ for all } n \geq 0.$$

- iii *For any signed measure* $\mu \in \mathcal{B}$ *the energy integral is non-negative:* $I_F(\mu) \geq 0$.
- iv *There exists a function* $f \in L^2_{w_{(d-1)/2}}[-1,1]$ *such that*

$$F(x \cdot y) = \int_{\mathbb{S}^d} f(x \cdot z) f(z \cdot y)\, d\sigma(z), \quad x, y \in \mathbb{S}^d.$$

- v $I_F(\mu) \geq I_F(\sigma) \geq 0$ *for any Borel probability measure* μ.

Generalized Stolarsky principle

Define the L^2 discrepancy of a Borel probability measure μ w.r.t. the function $f : [-1,1] \to \mathbb{R}$ as

$$D^2_{L^2,f}(\mu) = \int\limits_{\mathbb{S}^d} \left| \int\limits_{\mathbb{S}^d} f(x \cdot y) d\mu(y) - \int\limits_{\mathbb{S}^d} f(x \cdot y) d\sigma(y) \right|^2 d\sigma(x).$$

In particular, if $\mu = \frac{1}{N} \sum \delta_{z_i}$

$$D^2_{L^2,f}(Z) = \int_{\mathbb{S}^d} \left(\frac{1}{N} \sum_{i=1}^{N} f(x \cdot z_i) - \int_{\mathbb{S}^d} f(x \cdot y) d\sigma(y) \right)^2 d\sigma(x).$$

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$$D^2_{L^2,f}(\mu) = \int\limits_{\mathbb{S}^d} \left| \int\limits_{\mathbb{S}^d} f(x \cdot y) d\big(\mu - \sigma\big)(y) \right|^2 \, d\sigma(x).$$

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Theorem (DB, R. Matzke, F. Dai, '18)

Generalized Stolarsky principle:

Let F be positive definite and f as in (iv), *then*

$$I_F(\mu) - I_F(\sigma) = D^2_{L^2,f}(\mu).$$

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- $I_F(\mu) - I_F(\sigma) = I_F(\mu - \sigma).$

Discrepancy/energy bounds

Theorem (DB, F. Dai, '19)

Assume that F is positive definite and f as in (iv)*.*

Let $Z = \{z_1, ..., z_N\} \subset \mathbb{S}^d$ *and* $\mu = \frac{1}{N}\sum_{i=1}^{N} \delta_{z_i}$.

- `Upper bound:`

$$\inf_{\#Z=N} D^2_{L^2,f}(\mu) \lesssim \frac{1}{N} \max_{0 \le \theta \lesssim N^{-\frac{1}{d}}} \big(F(1) - F(\cos\theta)\big).$$

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- Upper bound:

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- Lower bound:

$$\inf_{\#Z=N} D^2_{L^2,f}(\mu) \gtrsim \min_{1 \le k \lesssim N^{1/d}} \widehat{F}(k,\lambda).$$

Stolarsky principle on metric spaces

- Ω compact metric space
- $K : \Omega \times \Omega \to \mathbb{R}$ continuous, symmetric, and *positive definite*, i.e. $\sum_{i,j=1}^{n} K(x_i, x_j) c_i c_j \geq 0$.
- Define $T_K f(x) = \int_\Omega K(x, y) f(y) d\mu(y)$
- $T_K \phi_i = \lambda_i \phi_i$, $\lambda_i \geq 0$.

Theorem (Mercer)

$$K(x, y) = \sum_{i=1}^{\infty} \lambda_i \phi_i(x) \phi_i(y),$$

where the sum above converges absolutely and uniformly.

Stolarsky principle on metric spaces

Theorem (DB, O. Vlasiuk, '19?)

Let $K : \Omega \times \Omega \to \mathbb{R}$ *continuous, symmetric, and positive definite. Let* $\tilde{\mu}$ *be the equilibrium measure of* I_K *and* $\operatorname{supp}\mu \subset \operatorname{supp}\tilde{\mu}$.
Then there exists $f : \widetilde{\Omega} \times \widetilde{\Omega} \to \mathbb{R}$ *such that*

$$I_K(\mu) - I_K(\tilde{\mu}) = D^2_{L^2,f,\tilde{\mu}}(\mu),$$

where

$$D^2_{L^2,f,\tilde{\mu}}(\mu) = \int_\Omega \left| \int_\Omega f(x,y)d\mu(y) - \int_\Omega f(x,y)d\tilde{\mu}(y) \right|^2 d\tilde{\mu}(x).$$

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Let $K : \Omega \times \Omega \to \mathbb{R}$ continuous, symmetric, and positive definite. Let $\tilde{\mu}$ be the equilibrium measure of I_K and $\operatorname{supp} \mu \subset \operatorname{supp} \tilde{\mu}$.
Then there exists $f : \widetilde{\Omega} \times \widetilde{\Omega} \to \mathbb{R}$ such that

$$I_K(\mu) - I_K(\tilde{\mu}) = D^2_{L^2, f, \tilde{\mu}}(\mu),$$

where

$$D^2_{L^2, f, \tilde{\mu}}(\mu) = \int_{\Omega} \left| \int_{\Omega} f(x,y) d\mu(y) - \int_{\Omega} f(x,y) d\tilde{\mu}(y) \right|^2 d\tilde{\mu}(x).$$

$$f(x,y) = \sum_{i=1}^{\infty} \sqrt{\lambda_i}\, \phi_i(x) \phi_i(y).$$

Stolarsky principle for hemispheres

Theorem (DB, Dai, Matzke '18; Skriganov '18)

$$D^2_{L^2,\text{hem}}(Z) = \frac{1}{2\pi}\left(\int\limits_{\mathbb{S}^d}\int\limits_{\mathbb{S}^d} d(x,y)\, d\sigma(x)\, d\sigma(y) - \frac{1}{N^2}\sum_{i,j=1}^{N} d(z_i, z_j) \right)$$

Stolarsky principle for hemispheres

Theorem (DB, Dai, Matzke '18; Skriganov '18)

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Corollary (DB, Dai, Matzke '18)

For any $Z = \{z_1, \dots, z_N\} \subset \mathbb{S}^d$

$$\frac{1}{N^2}\sum_{i,j=1}^{N} d(z_i, z_j) \le \frac{\pi}{2}$$

Stolarsky principle for hemispheres

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Corollary (DB, Dai, Matzke '18)

For any $Z = \{z_1, ..., z_N\} \subset \mathbb{S}^d$

$$\frac{1}{N^2}\sum_{i,j=1}^{N} d(z_i, z_j) \le \frac{\pi}{2}$$

For even N*:*

$$\frac{1}{N^2}\sum_{i,j=1}^{N} d(z_i, z_j) = \frac{\pi}{2} \Longleftrightarrow Z \text{ - symmetric.}$$

Simple corollaries: a conjecture of Fejes Tóth

Corollary (DB, Dai, Matzke '18)

For odd N:

$$\frac{1}{N^2}\sum_{i,j=1}^{N} d(z_i,z_j) \le \frac{\pi}{2} - \frac{\pi}{2N^2}.$$

Maximum is achieved if and only if $Z = Z_1 \cup Z_2$, where

- *Z_1 is symmetric,*
- *Z_2 lies on a two-dimensional hyperplane (a great circle) and is a maximizer for $\mathbb{S}^1$.*

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- Fejes-Toth '59: $d = 1$ and conjectured for $d \geq 2$.
- Sperling, '60 ($d = 2$, even N)
- Larcher, '61 ($d = 2$, odd N, ?)
- Nielsen, '65 ($d = 2$)
- Kelly, '69 ($d \geq 2$)

Let $H(x) = C(x, 0)$ denote the hemisphere with center at x and let μ be a Borel probability measure on $\mathbb{S}^d$.

$$\int_{\mathbb{S}^d} \left(\mu\big(H(x)\big) - \frac{1}{2} \right)^2 d\sigma(x) = \frac{1}{2\pi} \cdot \left(\frac{\pi}{2} - \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} d(x, y)\, d\mu(x) d\mu(y) \right).$$

Hemisphere Stolarsky for general measures

Let $H(x) = C(x,0)$ denote the hemisphere with center at x and let μ be a Borel probability measure on $\mathbb{S}^d$.

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- For any probability measure μ: $\quad I_{\text{geod}}(\mu) \le \frac{1}{2}$.

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Let $H(x) = C(x, 0)$ denote the hemisphere with center at x and let μ be a Borel probability measure on $\mathbb{S}^d$.

$$\int_{\mathbb{S}^d} \left(\mu(H(x)) - \frac{1}{2} \right)^2 d\sigma(x) = \frac{1}{2\pi} \cdot \left(\frac{\pi}{2} - \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} d(x,y)\, d\mu(x) d\mu(y) \right).$$

- For any probability measure μ: $\quad I_{\text{geod}}(\mu) \leq \frac{1}{2}$.
- $I_{\text{geod}}(\mu) = \frac{1}{2}$ (i.e. μ is a maximizer) $\quad$ iff $\mu(H(x)) = \frac{1}{2}$ for σ-a.e. $x \in \mathbb{S}^d$

Hemisphere Stolarsky for general measures

Let $H(x) = C(x,0)$ denote the hemisphere with center at x and let μ be a Borel probability measure on $\mathbb{S}^d$.

$$\int_{\mathbb{S}^d} \left(\mu(H(x)) - \frac{1}{2} \right)^2 d\sigma(x) = \frac{1}{2\pi} \cdot \left(\frac{\pi}{2} - \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} d(x,y)\, d\mu(x) d\mu(y) \right).$$

- For any probability measure μ: $\quad I_{\text{geod}}(\mu) \leq \frac{1}{2}$.
- $I_{\text{geod}}(\mu) = \frac{1}{2}$ (i.e. μ is a maximizer) iff
  $\mu(H(x)) = \frac{1}{2}$ for σ-a.e. $x \in \mathbb{S}^d$ iff
  μ is symmetric, i.e. $\mu(E) = \mu(-E)$.

Geodesic distance energy integrals

Let μ be a Borel probability measure on the sphere $\mathbb{S}^d$. For $\lambda > 0$ define the energy integral

$$I_\lambda(\mu) = \int\limits_{\mathbb{S}^d}\int\limits_{\mathbb{S}^d} \big(d(x,y)\big)^\lambda d\mu(x) d\mu(y)$$

Maximizers (DB, Dai, Matzke '18):

- $0 < \lambda < 1$: unique maximizer is σ,
- $\lambda = 1$: any symmetric measure,
- $\lambda > 1$: mass $\frac{1}{2}$ at two opposite poles.

Euclidean distance energy integrals

Let μ be a Borel probability measure on the sphere $\mathbb{S}^d$. For $\lambda > 0$ define the energy integral

$$I_\lambda(\mu) = \int_{\mathbb{S}^d}\int_{\mathbb{S}^d} \|x-y\|^\lambda d\mu(x) d\mu(y)$$

Maximizers (Björck '56):

- $0 < \lambda < 2$: unique maximizer is surface measure,
- $\lambda = 2$: any measure with center of mass at 0,
- $\lambda > 2$: mass $\frac{1}{2}$ at two opposite poles.

Another conjecture of Fejes Tóth

Conjecture (Sum of acute angles)

Let $Z = \{z_1, ..., z_N\} \subset \mathbb{S}^d$ *and define*

$$F(x \cdot y) = \min \big\{ \arccos(x \cdot y), \, \pi - \arccos(x \cdot y) \big\} = \arccos |x \cdot y|,$$

i.e. the acute angle between the lines through x, $y \in \mathbb{S}^d$.

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- $\max I_F(\mu) = I_F(\nu_{ONB}) = \frac{\pi}{2} \cdot \frac{d}{d+1}$, *where*

$$\nu_{ONB} = \frac{1}{d+1}\sum_{i=1}^{d+1} \delta_{e_i}$$

Acute angles: known results

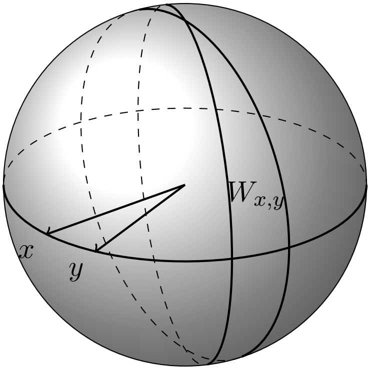
- Partial results for $d \geq 2$

  Fodor, Vigh, Zarnocz, '16; DB, R. Matzke, '18

- Known on $\mathbb{S}^1$
- Stolarsky-type proof: Quadrant discrepancy

$$\sigma\big(Q(x) \cap Q(y)\big) = \frac{1}{2} - \frac{1}{\pi} \arccos |x \cdot y|$$

$$D^2_{L^2,quad}(Z) = \frac{1}{4} - \frac{1}{\pi} E_F(Z)$$



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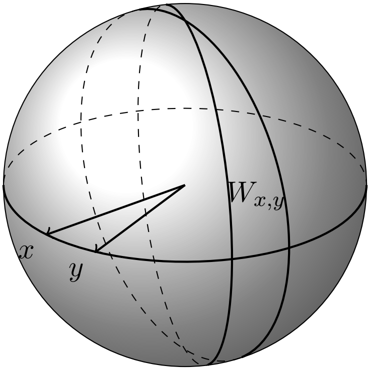
$$W_{xy} = H_x \triangle H_y$$
$$= \{z \in \mathbb{S}^d : \mathrm{sign}\langle z, x\rangle \neq \mathrm{sign}\langle z, y\rangle\}$$

Tessellations and discrepancy



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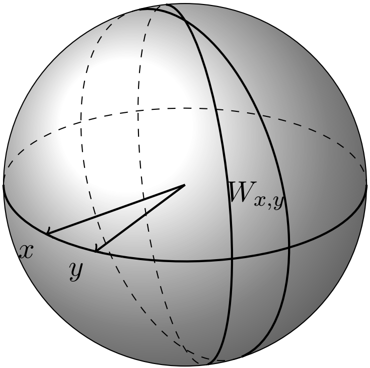

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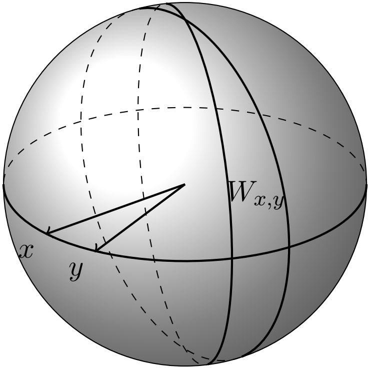

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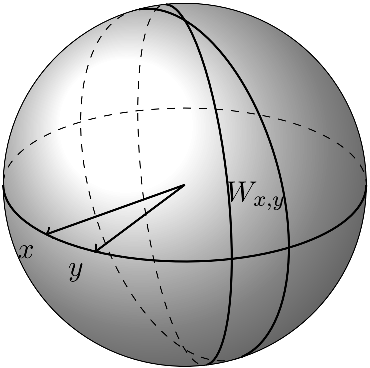

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$$\Delta_Z(x, y) = d_H(x, y) - d(x, y) = \frac{\#(Z \cap W_{xy})}{N} - \sigma(W_{xy})$$

Stolarsky principle for wedge discrepancy (DB, Lacey)

Define the L^2 discrepancy for wedges

$$\|\Delta_Z(x,y)\|_2^2 = \int_{\mathbb{S}^d}\int_{\mathbb{S}^d}\left(\frac{1}{N}\sum_{k=1}^{N}\mathbf{1}_{W_{xy}}(z_k)-\sigma(W_{xy})\right)^2 d\sigma(x)\,d\sigma(y)$$

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Theorem (Stolarsky principle for the tessellation of the sphere)

For any finite set $Z = \{z_1, \dots, z_N\} \subset \mathbb{S}^d$

$$\left\|\Delta_Z(x,y)\right\|_2^2 = \frac{1}{N^2}\sum_{i,j=1}^{N}\left(\frac{1}{2} - d(z_i,z_j)\right)^2 - \int\limits_{\mathbb{S}^d}\int\limits_{\mathbb{S}^d}\left(\frac{1}{2} - d(x,y)\right)^2 d\sigma(x)\,d\sigma(y).$$

Finite metric spaces (Barg)

- X - finite metric space, $Z = \{z_1, ..., z_N\} \subset X$
- $D^2_{L^2}(Z) = \sum_{t=0}^{n} \sum_{x \in X} \left(\frac{1}{N} \sum \mathbf{1}_{B(x,t)}(z_j) - \frac{\#B(x,t)}{\#X} \right)^2$
- Define the distance

$$\lambda(x,y) = \frac{1}{2} \sum_{u \in X} \big| d(x,u) - d(y,u) \big|.$$

- Stolarsky principle:

$$D^2_{L^2}(Z) = \frac{1}{(\#X)^2} \sum_{x,y \in X} \lambda(x,y) - \frac{1}{N^2} \sum_{i,j=1}^{N} \lambda(z_i, z_j).$$

Hamming cube (Barg)

$$D^2_{L^2}(Z) = \frac{1}{(\#X)^2} \sum_{x,y \in X} \lambda(x,y) - \frac{1}{N^2} \sum_{i,j=1}^{N} \lambda(z_i, z_j).$$

Lemma

Let $X = \{0,1\}^n$ *be the Hamming cube. Assume that the Hamming distance* $d(x,y) = w$. *Then*

$$\lambda(x,y) = 2^{n-w} w \binom{w-1}{\lceil \frac{w}{2} \rceil - 1}.$$