Frame Design Using Projective Riesz Energy

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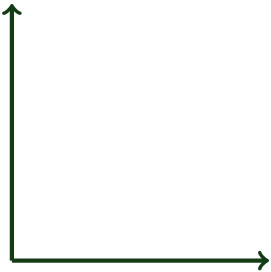
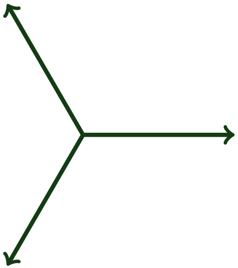
This is joint work with Doug Hardin and Ed Saff.

Frames


a basis


a frame

Frames

A set of vectors $X = \{x_i\}_{i \in I}$ is a frame for a separable Hilbert space H if there exist $A, B > 0$ such that for every $x \in H$,

$$A\|x\|^2 \leq \sum_{i \in I} |\langle x, x_i \rangle|^2 \leq B\|x\|^2.$$

The constants A, B are called the frame bounds.

When $A = B$, X is called a tight frame, which generalizes the concept of an orthonormal basis in the sense that we have the following recovery formula

$$x = \frac{1}{A} \sum_{i \in I} \langle x, x_i \rangle x_i, \quad \forall x \in H.$$

Desirable properties of frames

equal norm tight symmetry well separation	$\rightarrow$	robustness to erasures stability in reconstruction	$\rightarrow$	signal processing coding theory quantum inf. theory

[1] R. B. Holmes and V. I. Paulsen. Optimal frames for erasures. Linear Algebra Appl. 377 (2004): 31-51
[2] J. M. Renes, R. Blume-Kohout, A. J. Scott, and C. M. Caves. Symmetric informationally complete quantum measurements. J. Math. Phys., 45(6):2171–2180, 2004.
[3] T. Strohmer and R. W. Heath, Jr. Grassmannian frames with applications to coding and communication. Appl. Comput. Harmon. Anal., 14(3):257–275, 2003.

Desirable properties of frames

$\mathcal{S}(d, N)$ = the set of all unit norm frames (with N frame vectors) for $\mathbb{H}^d$
= the collection of all N-point configurations on $\mathbb{S}^{d-1}$

We will focus on finding unit norm finite frames $X = [x_1, \cdots, x_N]$ that are

- nearly tight: $XX^* \approx \frac{N}{d} I_d$
- well-separated: coherence $\xi(X) = \max_{i \neq j} |\langle x_i, x_j \rangle|$ is small.

$\min_{X \in \mathcal{S}(d,N)} \xi(X)$ is the best line-packing problem [4].

[4] J. H. Conway, R. H. Hardin, and N. J. A. Sloane. Packing lines, planes, etc.: Packings in Grassmannian spaces. Exp. Math. 5.2 (1996)

Desirable properties of frames

A frame $X = \{x_i\}_{i=1}^{N}$ is an equiangular tight frame (ETF) if

- X is tight, and
- $\frac{|\langle x_i, x_j \rangle|}{\|x_i\|_2 \|x_j\|_2}$ is constant for any pair $i \neq j$.

ETF is exactly tight and best-separated, but it does not always exist.

[5] M. Fickus and D. G. Mixon. Tables of the existence of equiangular tight frames.
[6] S. Waldron. An introduction to finite tight frames. Boston: Birkhäuser, 2018.

Previous work

Minimizing frame potential [7]

$$\min_{X \in \mathcal{S}(d,N)} \sum_{i \neq j} |\langle x_i, x_j \rangle|^2 \tag{1}$$

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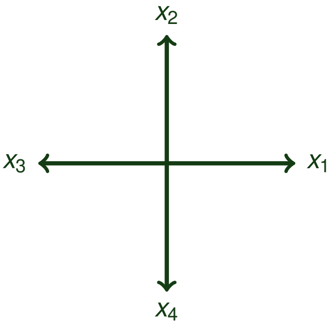
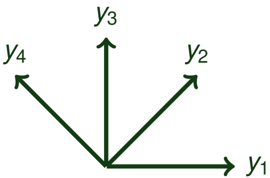
The minimizing configurations of (1) are exactly unit norm tight frames.



Not well separated



well separated

[7] J. Benedetto and M. Fickus. Finite normalized tight frames. Adv. Comput. Math. 2003

Previous work

$$\min_{X \in \mathcal{S}(d,N)} \sum_{i \neq j} |\langle x_i, x_j \rangle|^p \tag{2}$$

- Bigger p value promotes separation, since (2) approaches coherence minimizing as $p \to \infty$.
- If $p > 2$, ETF (when exists) is the minimizer.

[8] M. Ehler and K. Okoudjou. Minimization of the Probabilistic *p*-frame Potential, J. Statist. Plann. Inference, 142 (2012)
[9] D. Bilyk, D., A. Glazyrin, R. Matzke, J. Park, and O. Vlasiuk. Optimal measures for p-frame energies on spheres. (2019)
[10] X. Chen, V. Gonzalez, E. Goodman, S. Kang, and K. Okoudjou. Universal optimal configurations for the p-frame potentials. Adv. Comput. Math (2020)

Main result

Solving $\min_{X \in \mathcal{S}(d,N)} \sum_{i \neq j} \frac{1}{\left(\sqrt{2 - 2|\langle x_i, x_j \rangle|^2}\right)^s}$ produces nearly tight and well-separated (low coherence) frames.

$$\sqrt{2 - 2|\langle x, y \rangle|^2} = \|xx^* - yy^*\|_F$$

Set up

$$\min_{X \in \mathcal{S}(d,N)} \sum_{i \neq j} f\left(\sqrt{2 - 2|\langle x_i, x_j \rangle|^2}\right)$$

where f is decreasing and convex on $(0, \sqrt{2}]$.

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$f(t) = \frac{1}{t^s}, s > 0$ decreasing, strictly convex	$\min_{X \in \mathcal{S}(d,N)} \sum_{i \neq j} \frac{1}{\left(\sqrt{2 - 2\|\langle x_i, x_j \rangle\|^2}\right)^s}$
$f(t) = -\log(t)$ decreasing, strictly convex	$\min_{X \in \mathcal{S}(d,N)} \sum_{i \neq j} \log \frac{1}{\sqrt{2 - 2\|\langle x_i, x_j \rangle\|^2}}$
$f(t) = \left(\frac{2 - t^2}{2}\right)^{p/2}, p > 0$ decreasing, f is not convex	$\min_{X \in \mathcal{S}(d,N)} \sum_{i \neq j} \|\langle x_i, x_j \rangle\|^p$

Set up

$\mathcal{S}(d, N)$ = the collection of all N-point configurations on $\mathbb{S}^{d-1}$
$\mathcal{P}(d, N)$ = the collection of all N-point configurations on $\mathbb{P}^{d-1}$

$X = \{x_i\}_{i=1}^N$
$P = \{p_i = x_i x_i^*\}_{i=1}^N$

$$\min_{X \in \mathcal{S}(d,N)} \sum_{i \neq j} f\left(\sqrt{2 - 2|\langle x_i, x_j \rangle|^2}\right) = \min_{P \in \mathcal{P}(d,N)} \sum_{i \neq j} f(\|p_i - p_j\|)$$

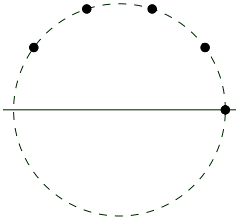
Results on $\mathbb{R}^2$

Theorem 1 (C. Hardin, Saff, 2020)

If f is a non-increasing convex function, then equally distributed points on half circle is an optimal configuration of

$$\min_{X \in \mathcal{S}(d,N)} \sum_{i \neq j} f\left(\sqrt{2 - 2|\langle x_i, x_j \rangle|^2}\right)$$

If, in addition, f is strictly convex, then no other N-point configuration is optimal.



tight

best-separated

Results related to ETF

Theorem 2 (C., Hardin, Saff, 2020)

Let f be decreasing and strictly convex, then ETF (when exists) is the unique minimizer of

$$\min_{X \in \mathcal{S}(d,N)} \sum_{i \neq j} f\left(\sqrt{2 - 2|\langle x_i, x_j \rangle|^2}\right).$$

Results related to ETF

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Theorem 2 (C., Hardin, Saff, 2020)

This is a stronger theorem:
Let g be decreasing and strictly convex, then ETF (when exists) is the unique minimizer of

$$\min_{X\in\mathcal{S}(d,N)} \sum_{i\neq j} g\left(2-2|\langle x_i, x_j\rangle|^2\right).$$

$|\langle x, y\rangle|^p = g(2-2|\langle x, y\rangle|^2)$, where $g(t) = \left(\frac{2-t}{2}\right)^{p/2}$ is decreasing and strictly convex when $p > 2$.

Results related to ETF

Proof of Theorem 2:
Let $p_i = x_i x_i^*$.

$$J := \sum_{i\neq j} \|p_i - p_j\|^2 = \sum_{i=1}^{N} \sum_{j\neq i} (2 - 2\langle p_i, p_j\rangle)$$
$$= \sum_{i=1}^{N} \left(2(N-1) - 2\sum_{j=1}^{N} \langle p_i, p_j\rangle + 2 \right)$$
$$= 2N^2 - 2\sum_{i,j=1}^{N} |\langle x_i, x_j\rangle|^2 \leq 2N^2 - 2N^2/d,$$

$$E(X) = \frac{N(N-1)}{1} \cdot \frac{1}{N(N-1)} \sum_{i\neq j} g(\|p_i - p_j\|^2)$$
$$\geq \frac{N(N-1)}{1} g\left(\sum_{i\neq j} \frac{1}{N(N-1)} \|p_i - p_j\|^2 \right) = N(N-1) g\left(\frac{1}{N(N-1)} J \right)$$
$$\geq N(N-1) g\left(\frac{2N^2 - 2N^2/d}{N(N-1)} \right) = N(N-1) g\left(\frac{2N(1-1/d)}{(N-1)} \right).$$

Projective Riesz kernel

$$\frac{1}{\left(\sqrt{2-2|\langle x_i, x_j\rangle|^2}\right)^s} = \frac{1}{\|p_i - p_j\|^s}$$

Theorem 3 (Separation)

Let $s > \alpha = \dim(\mathbb{P}^{d-1})$. If X_N is an N-point minimizing configuration of

$$\min_{X \in \mathcal{S}(d,N)} \sum_{i \neq j} \frac{1}{\left(\sqrt{2-2|\langle x_i, x_j\rangle|^2}\right)^s}, \tag{3}$$

then

$$\xi(X_N) \leq \sqrt{1 - \frac{C^2}{2} N^{-2/\alpha}},$$

where the constant C depends on s and d.

Projective Riesz kernel

Proof uses a result from [11]

Suppose $A \subset \mathbb{R}^m$ is compact and supports an upper α-regular measure μ. Let $s > \alpha, N \geq 2$ be fixed. If P_N is an N-point minimizing configuration on A for the s-Riesz energy minimizing problem, then

$$\delta(P_N) \geq C_1 N^{-\frac{1}{\alpha}},$$

where $C_1 = \left(\frac{\mu(A)}{C_A} (1 - \frac{\alpha}{s}) \right)^{1/\alpha} \left(\frac{\alpha}{s} \right)^{\frac{1}{s}}$.

$$\delta^2(P_N) = \min_{i \neq j} \|p_i - p_j\|^2 = \min_{i \neq j} (2 - 2|\langle x_i, x_j \rangle|^2) = 2 - 2\xi^2(X_N)$$

[11] D. P. Hardin, E. B. Saff, and J. T. Whitehouse. Quasi-uniformity of minimal weighted energy points on compact metric spaces. J. Complexity 28.2 (2012)

Projective Riesz kernel

Theorem 4 (Nearly tight)

Let $s \geq 0$. If X_N is an N-point minimizing configuration of

$$\min_{X \in \mathcal{S}(d,N)} \sum_{i \neq j} \frac{1}{\left(\sqrt{2 - 2|\langle x_i, x_j \rangle|^2}\right)^s},$$

then

$$\lim_{N \to \infty} \frac{1}{N} X_N X_N^* = \frac{1}{d} I_d. \tag{4}$$

Projective Riesz kernel

Theorem 4 (Nearly tight)

Let $s \geq 0$. If X_N is an N-point minimizing configuration of

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then

$$\lim_{N \to \infty} \frac{1}{N} X_N X_N^* = \frac{1}{d} I_d. \qquad (4)$$

The counting measure converges weak* to the uniform measure.

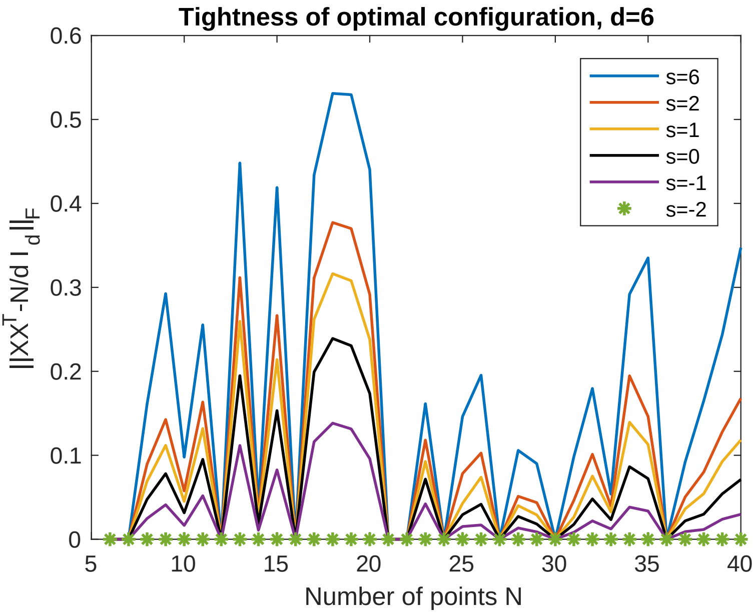
Projective Riesz kernel

Theorem 4 says that the optimal configurations are nearly tight asymptotically. A more desirable result would be

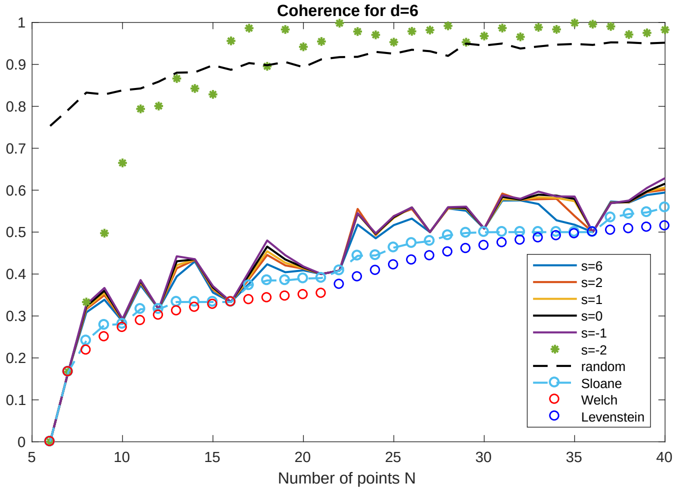
$$\left\| X_N X_N^* - \frac{N}{d} I_d \right\|_F = \mathcal{O}(N^{-q}), \quad q > 0. \tag{5}$$

Numerical experiments suggest this holds for small values of s.

Numerical experiment: tightness



Numerical experiment: separation



Numerical experiment: a proposal

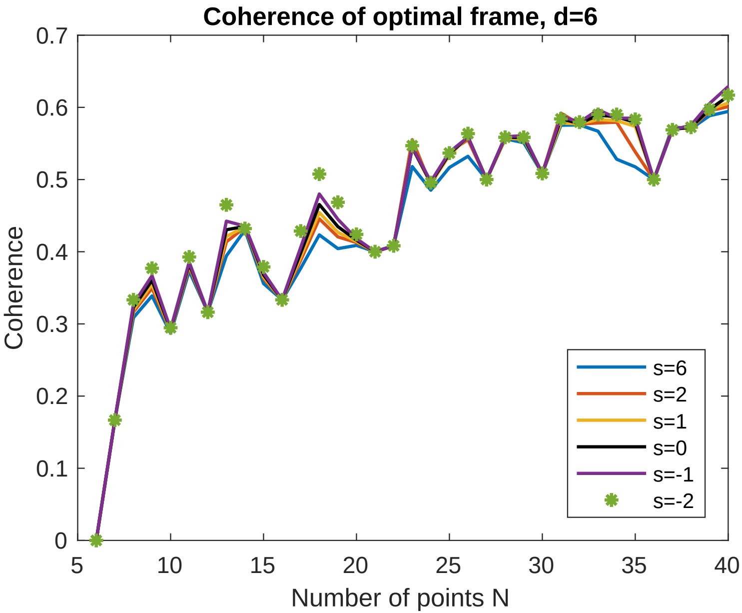
A method to generate well separated tight frame:

1. Generate a random frame $X \in \mathcal{S}(d, N)$.
2. Using X as an initial configuration, use an optimization algorithm (such as gradient descent) to find a local minimizer Y for

$$\min_{X \in \mathcal{S}(d,N)} \sum_{i \neq j} \frac{1}{\left(\sqrt{2 - 2|\langle x_i, x_j \rangle|^2}\right)^s} \tag{3}$$

   for some $s > d - 1$. Y is expected to be well-separated and nearly tight according to our theorems.
3. Solve $\min_{X \in \mathcal{S}(d,N)} \sum_{i \neq j} |\langle x_i, x_j \rangle|^2$ using Y as the initial configuration.

Numerical experiment: a proposal



Thank you

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