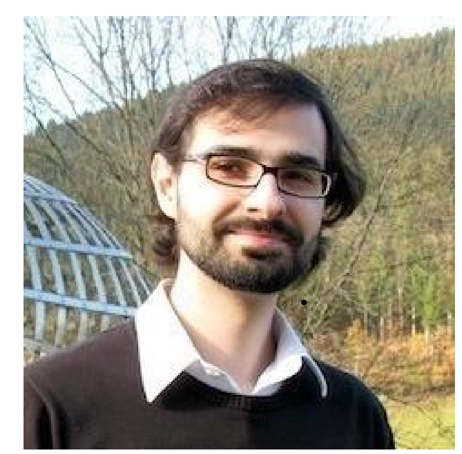
Semidefinite Programming Bounds for the Average Kissing Number

Maria Dostert (EPFL)

joint work with



Alexander Kolpakov
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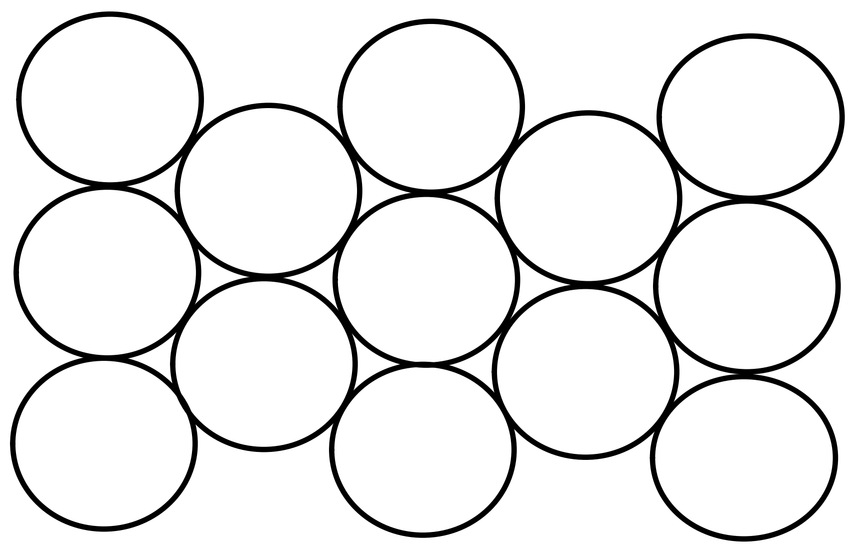
(Delft)

Average Kissing Number



Packing of unit disks by hexagonal lattice

Each disk touches 6 disks

$\Rightarrow$ Average kissing number of the hexagonal lattice is 6

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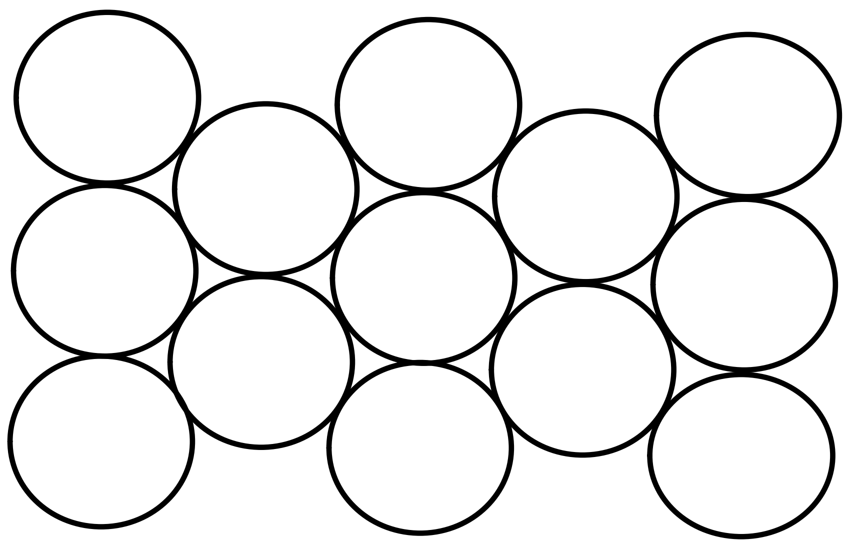

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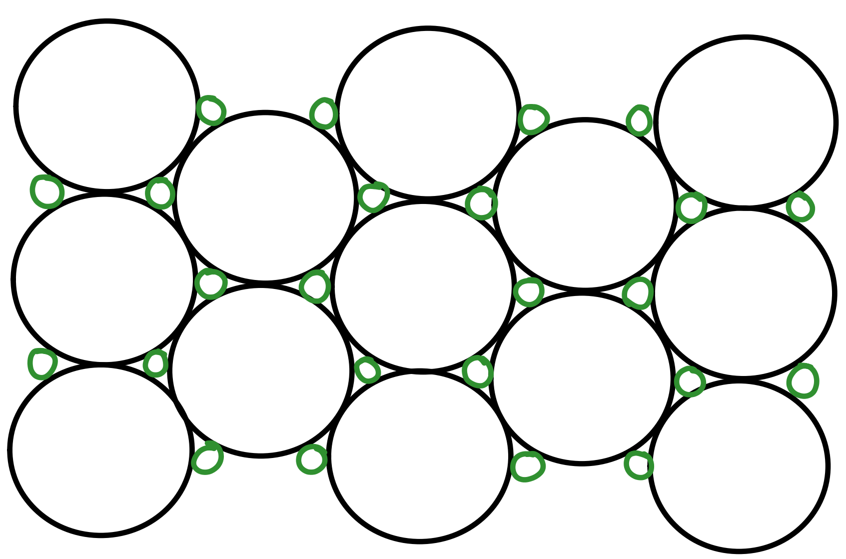

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here:
- any black disk touches 12 disks
- any green disk touches 3 disks
- the average is still 6

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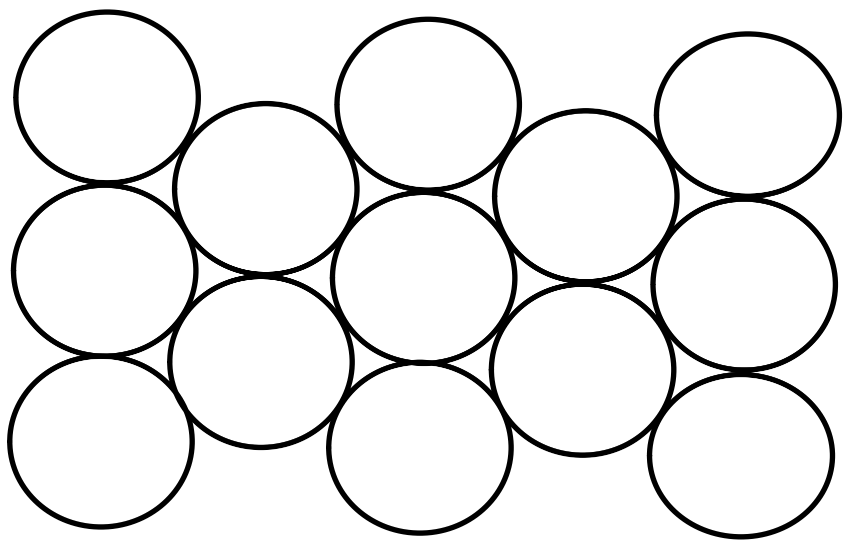

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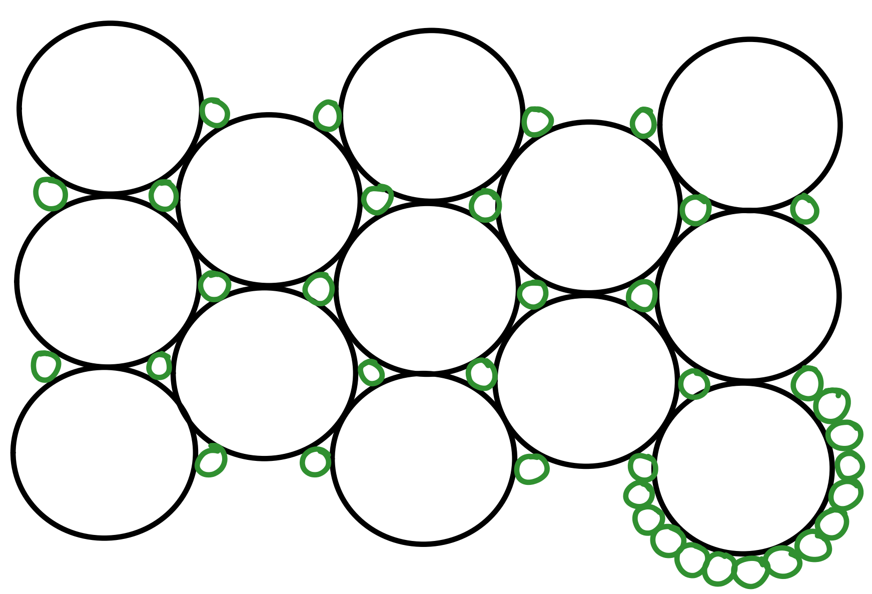

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Can we increase the average kissing number by adding more smaller disks?

Average Kissing Number

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Contact graphs of a packing $\mathcal{P}$: graph with vertex set $\mathcal{P}$ in which two balls X and Y are adjoint if $X \cap Y \neq \emptyset$

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Average kissing number in $\mathbb{R}^n$:

$$\boxed{k_n = \sup\{\bar{\delta}(G) : G \text{ is the contact graph of a packing of balls in } \mathbb{R}^n\}}$$

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Koebe-Andrew-Thurston: Contact graphs for packings of disks on the plane are <u>simple planar graphs</u> $\Rightarrow k_2 = 6$

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for $n>2$: problem unsolved.

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$$|E| \leq \sum_{X \in \mathcal{P}} |\{X, Y\} \in E : r(X) \leq r(Y)\}| \leq \tau_n \, |\mathcal{P}|$$

$$\bar{\delta}(G) = \frac{2|E|}{|\mathcal{P}|} \leq 2\tau_n \qquad \Rightarrow K_n \leq 2\tau_n$$

Upper Bound

First nontrivial upper bound by Kuperberg & Schramm

$\Rightarrow k_3 \leq 8 + 4\sqrt{3} = 14.928\ldots$

Glazyrin refines this approach: $k_3 \leq 13.955$

and extends it to higher dimension

beats $2\tau_n$ upper bound for $n = 4, 5$

Our goal: Refine Glazyrin's approach by using Semidefinite programming.

Notations

Euclidean inner product : $x \cdot y = \sum_{i=1}^{n} x_i y_i$, $x, y \in \mathbb{R}^n$

$(n-1)$ dim unit sphere $S^{n-1} = \{ x \in \mathbb{R}^n : \|x\| = 1 \}$

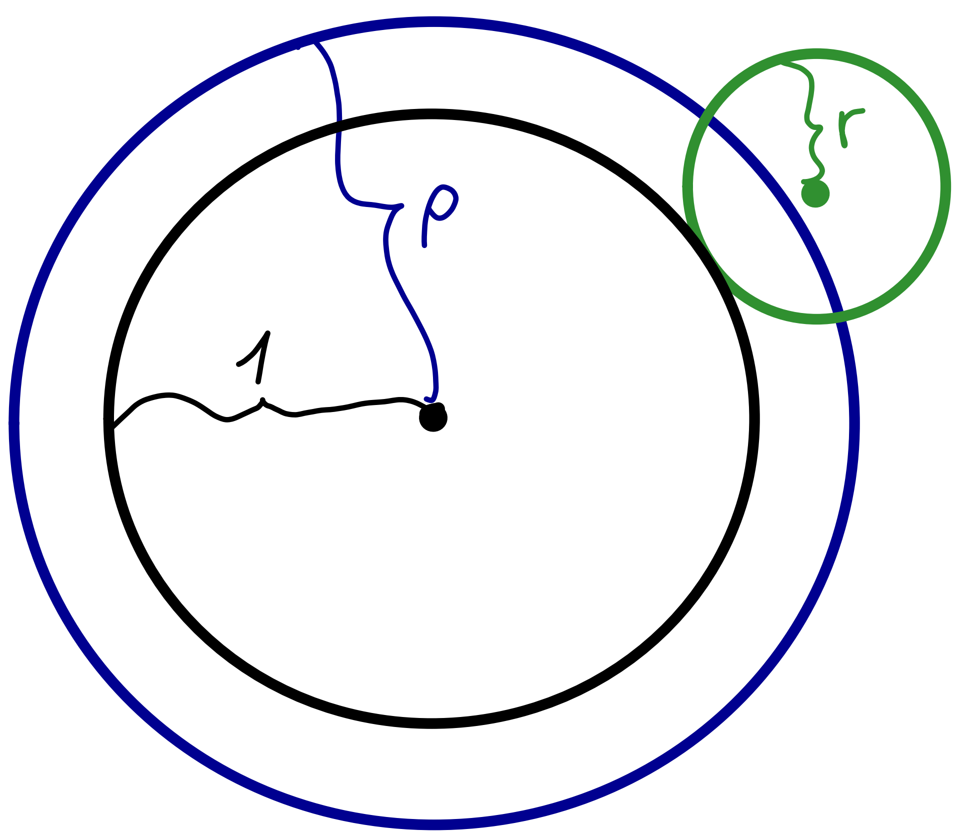
distance between $x, y \in S^{n-1}$: $\arccos x \cdot y$

surface measure of $(n-1)$ dim sphere of radius ρ : ω_ρ $(\omega = \omega_1)$

spherical cap in S^{n-1} of center $x \in S^{n-1}$ and radius α : $\{ y \in S^{n-1} : x \cdot y \geqslant \cos \alpha \}$

normalized area of this cap : $\dfrac{\omega(S^{n-2})}{\omega(S^{n-1})} \displaystyle\int_{\cos\alpha}^{1} (1-u^2)^{\frac{n-3}{2}} \, du$

Glazyrin's Upper Bound



Let $\rho > 1$, $r > 0$, dimension $n \geq 3$

B_r : ball of radius r tangent to B_1

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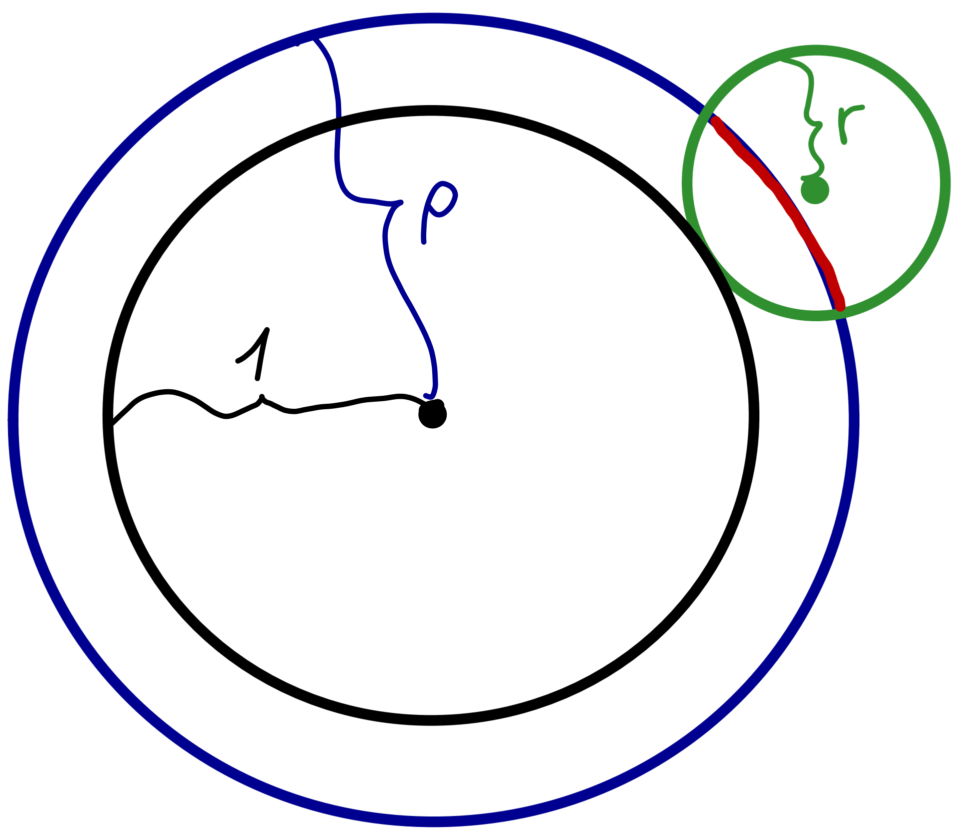

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$B_r \cap \rho S^{n-1}$ is either empty or a spherical cap.

Normalized area of this spherical cap:

$$A_{n,\rho}(r) = \frac{\omega_\rho(B_r \cap \rho S^{n-1})}{\omega_\rho(\rho S^{n-1})}$$

as a function of is monotonically increasing.

Glazyrin's Upper Bound

Lemma: If $n \geq 3$, $\rho > 1$, $r > 0$ then $A_{n,\rho}(r) + A_{n,\rho}\left(\frac{1}{r}\right) \geq 2 A_{n,\rho}(1)$

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Lemma: If $n \geq 3, \rho > 1, r > 0$ then $A_{n,\rho}(r) + A_{n,\rho}(\frac{1}{r}) \geq 2A_{n,\rho}(1)$

Fix $\rho > 1$ and consider unit ball at origin.

Any configuration of pairwise interior-disjoint balls tangent to central unit ball covers a fraction of ρS^{n-1} centered at origin.

dens_n: sup of covered fraction over all configurations

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For $\rho < 3$: $A_{n,\rho}(1) > 0$

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For $dens_n(\rho) \leq 1$, $\rho = \sqrt{3}$:

$k_3 \leq 14.928\ldots$ (Kuperberg & Schramm)

$\left.\begin{array}{l} k_4 \leq 34.680\ldots \\ k_5 \leq 77.756\ldots \end{array}\right\}$ (Glazyrin)

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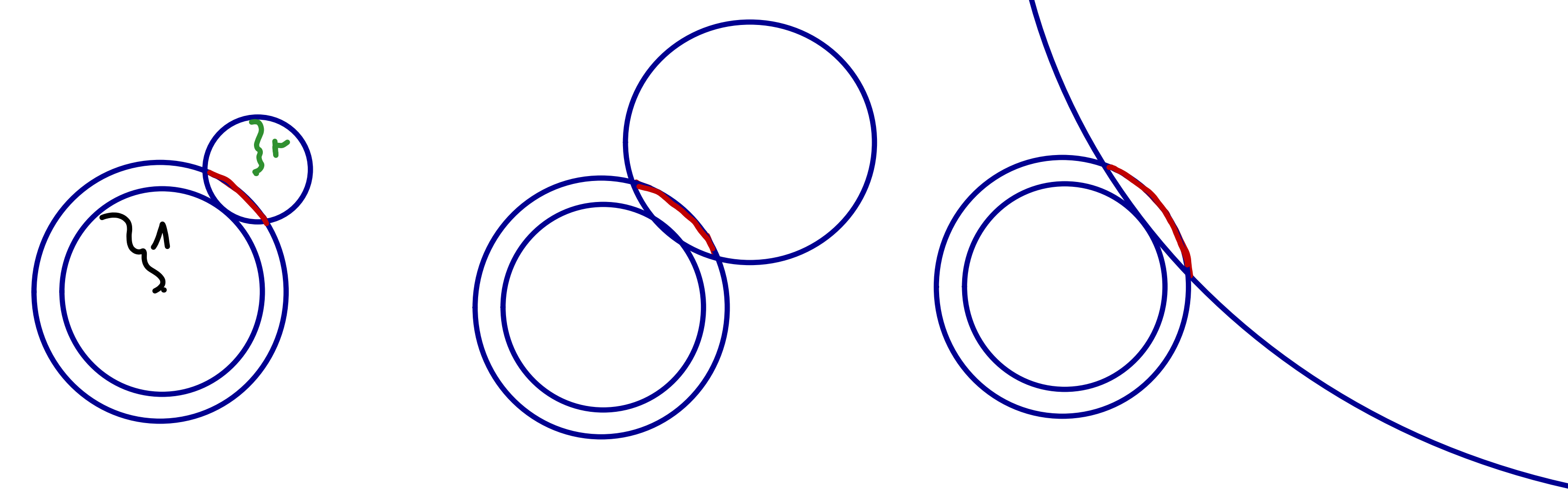
Glazyrin improved upper bound for $dens_3(\sqrt{3})$ $\Rightarrow k_3 \leq 13.955$.

Refining Glazyrin's approach using Semidefinite Programming

- $A_{n,\rho}$ increasing in r and has a limit $A_{n,\rho}(\infty)$



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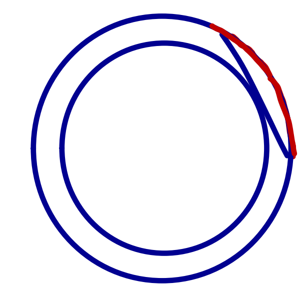

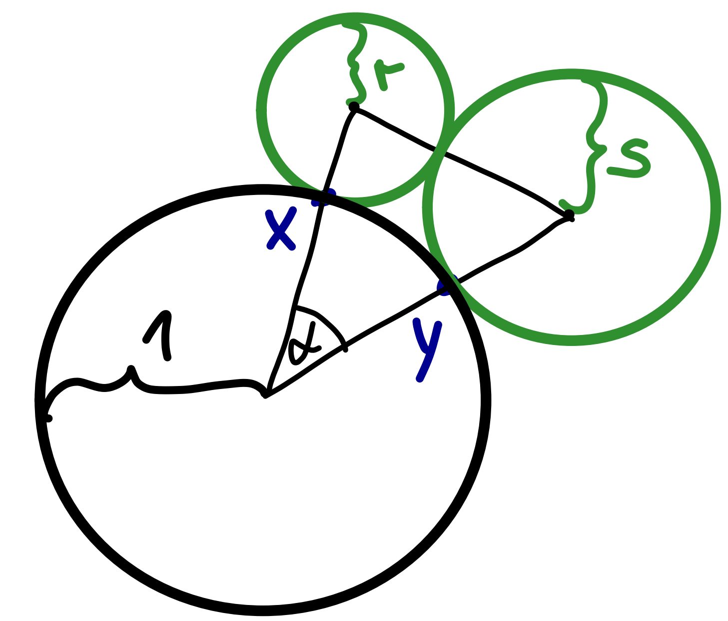

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$$\Rightarrow x \cdot y \leq \frac{1+r+s-rs}{1+r+s+rs} =: ip(r,s)$$

Refining Glazyrin's approach using Semidefinite Programming

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- kernel: real-valued square integrable function on $V\times V$ (V measure space)
- If $f: V \to \mathbb{R}$ is square integrable $\Rightarrow f\otimes f^*$ is the kernel of $(x,y) \mapsto f(x)f(y)$
- If $F: [0,1] \to \mathbb{R}$ is a kernel, $U \subseteq [0,1]$ finite, then $(F(u,v))_{u,v\in U}$ is a principal submatrix of F
- P_k^n: Jacobi polynomial degree k, $\alpha = \beta = \frac{(n-3)}{2}$, $P_k^n(1) = 1$

Semidefinite Programming Bound

Theorem : Let $n \geq 3$, $1 < p < 3$, R s.t. $R > \frac{p-1}{2}$, $r : [0,1] \to [\frac{p-1}{2}, R]$ increasing bijection

$a : [0,1] \to \mathbb{R}$ with $a(u) \geq A_{n,p}(r(u))^{1/2}$ for all $u \in [0,1]$

$a(1) \geq A_{n,p}(\infty)^{1/2}$

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Fix $d \geq 0$, for $k = 0, \dots, d$ let $F_k : [0,1]^2 \to \mathbb{R}$ be a kernel

$$f(t,u,v) = \sum_{k=0}^{d} F_k(u,v) P_k^n(t) \quad \text{for } t \in [-1,1],\ u,v \in [0,1]$$

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If 1) every principal submatrix of $F_0 - a \otimes a^* \succeq 0$,

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then

$$\boxed{\mathrm{dens}_n(p) \leq \max\{f(1,u,u) : u \in [0,1]\}}$$

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<u>Proof</u>: Δ normalized area of ρS^{n-1} covered by the configuration of $\mathcal{P}$

Assume $\Delta > 0$, assume each ball has radius $\geqslant \frac{\rho-1}{2}$

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B represented by (x, u), $x = B \cap$ central B_1

$I \subseteq S^{n-1} \times [0,1]$ set of pairs representing each $B \in \mathcal{P}$

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$$0 \leq \sum_{(x,u),(y,v) \in I} f(x \cdot y, u, v) a(u) a(v) - a(u)^2 a(v)^2 \leq \sum_{(x,u) \in I} f(1,u,u) a(u)^2 - \Big(\sum_{(x,u) \in I} a(u)^2 \Big)^2$$

$$\boxed{\Rightarrow \sum_{(x,u) \in I} a(u)^2 \leq \max \{ f(1,u,u) : u \in [0,1] \}}$$

How to specify the kernels $\mathcal{F}_k$?

- One way: Fix $N>0$ and functions $\rho_0, \dots, \rho_N : [0,1] \to \mathbb{R}$.

  Given $A \in \mathbb{R}^{(N+1)\times(N+1)}$, set

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$\Rightarrow$ We can rephrase SDP. Different choice of $p_0, \dots, p_N$ gives different SDP

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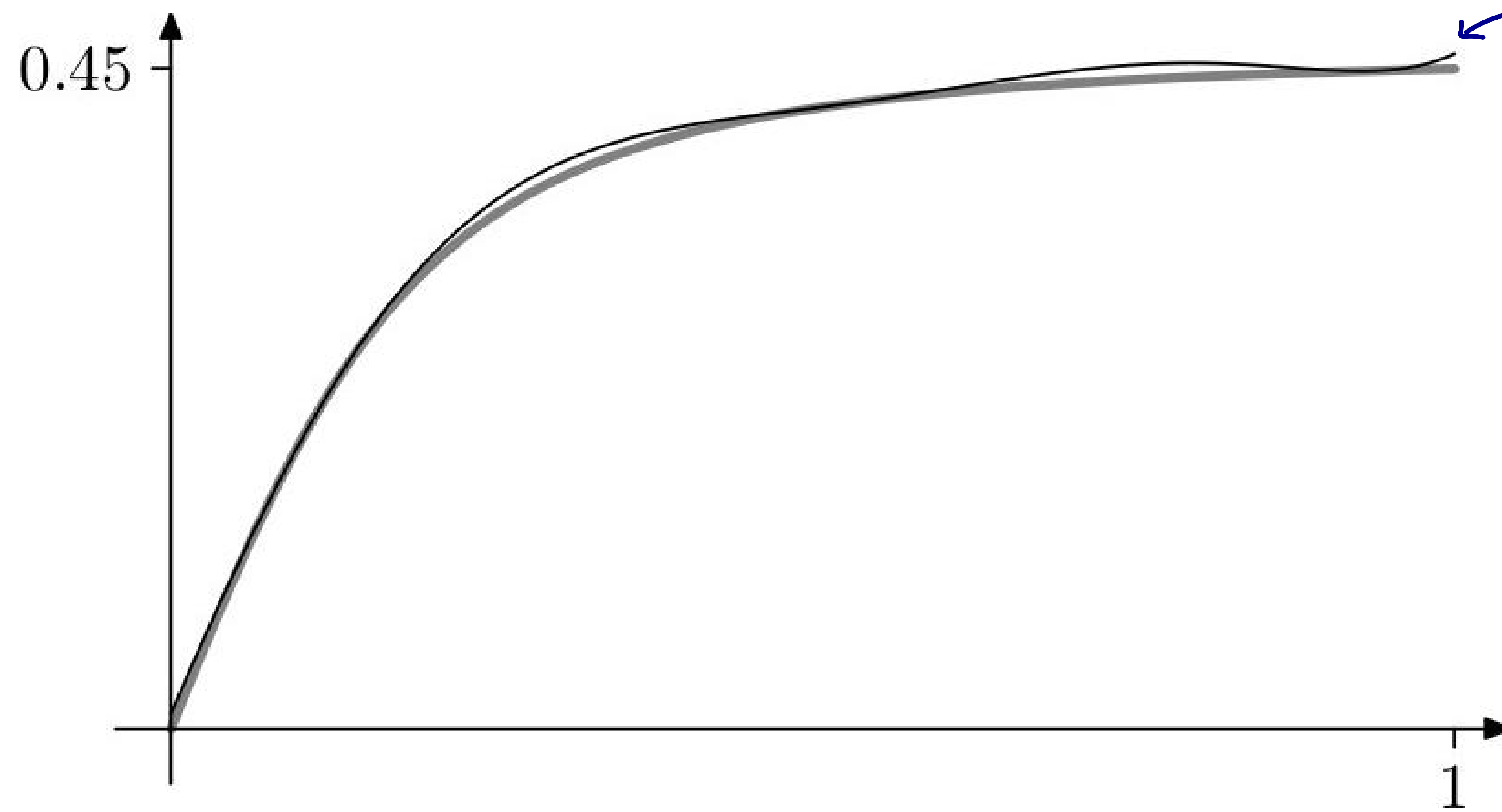
  Interplay between choice of p_i and quality of approximation a of $u\mapsto A_{n,p}(r(u))^{1/2}$

We used two approaches: 1) Set p_i as <u>step functions</u>

2) Take functions p_i to be <u>polynomials</u>

Approximation of $u \mapsto A_{n,p}(r(u))^{1/2}$



Jump to make $a(1) \geq A_{3,\sqrt{3}}(\infty)$

A polynomial of degree 6 (in black) that approximates $u \mapsto A_{3,\sqrt{3}}(r(u))^{1/2}$ (in gray) from above; here $R=30$

Results

DIMENSION	LOWER BOUND	PREVIOUS UPPER BOUND	NEW UPPER BOUND
3	12.612	13.955	13.606
4	24	34.681	27.439
5	40	77.757	64.022
6	72	156	121.105
7	126	268	223.144
8	240	480	408.386
9	272	726	722.629

SDP with polynomials $d=10$, $N=6,8$, $R=30$ (dimensions 3–4)

Step function approach $\rho=2$, $N=30$, $R\approx 184.25$ (dimensions 5–9)

Rigorous verification by Julia library of D., de Laat, Moustrou

Thank you!

Step function approach

Fix $R > \frac{\rho-1}{2}$, $r : [0,1] \to [\frac{\rho-1}{2}, R]$, $r(u) = (R - \frac{\rho-1}{2})u + \frac{\rho-1}{2}$

Fix $N > 0$ and points $0 = s_0 < s_1 < \ldots < s_N < s_{N+1} = 1$.

Let $S_i = [s_i, s_{i+1})$ for $i = 0, \ldots, N-1$, $S_N = [s_N, s_{N+1}]$

p_i is 1 on S_i , 0 otherwise.

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$$a(u) = \begin{cases} A_{n,\rho}(r(s_{i+1}))^{1/2} & \text{if } u \in S_i \text{ for some } i < N \\ A_{n,\rho}(\infty)^{1/2} & \text{if } u \in S_N \end{cases}$$

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$$\min \max \{ f_{ii}(1) : i = 0, \dots, N \}$$

$$f_{ij}(t) = \sum_{k=0}^{d} A_{k,ij} P_k^n(t),$$

$f_{ij}(t) \leq 0$ whenever $-1 \leq t \leq ip(r(s_i), r(s_j))$

$(A_{0,ij} - \alpha_i \alpha_j)_{i,j=0}^{N}$ is positive semidefinite,

$A_k \in \mathbb{R}^{(N+1)\times(N+1)}$ is positive semidefinite for $k = 0, \dots, d$.

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← By using Sampling

Polynomial approach

Fix $N>0$ and let $p_i(u) = u^i$ for $i = 0, \dots, N$

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Solve LP to get to obtain $\alpha_0,\dots\alpha_N$ (coefficients of $p_1,\dots p_N$)

Fix $\varepsilon>0$, and finite sample S : $a_0p_0(u)+\dots+a_Np_N(u)\geq A_{n,\rho}(r(u))^{1/2}+\varepsilon$ for sample points u

$$a_0p_0(1)+\dots+a_Np_N(1)\geq A_{n,\rho}(\infty)^{1/2}$$

Maximize $\quad \alpha_0p_0(u)+\dots+\alpha_N p_N(u)-A_{n,\rho}(r(u))^{1/2} : u\in S\}$

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we get approximation a and

use sum of squares to check $f(t,u,v)\leq 0$ for $\{(u,v,t): s_i(u,v,t)\geq 0 \quad i=1,\dots,4\}$

for some polynomials $s_1,\dots,s_4$

Polynomial approach

$\min \lambda$

$$f(t,u,v) = \sum_{k=0}^{d} \sum_{i,j=0}^{N} A_{k,ij}\, p_i(u) p_j(u) P_k^n(t)$$

$$f = -s_1 q_1 - s_2 q_2 - s_3 q_3 - s_4 q_4 - q_5$$

$$\lambda - f(1,u,u) = \ell_1(u) + (1-u)u\,\ell_2(u)$$

$q_1, \dots, q_5$ sum of squares polynomials in u, v, t

ℓ_1, ℓ_2 sum of squares polynomials in u

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