On point configurations and frame theory

Alex Iosevich

Point Distributions Webinar, February 2021

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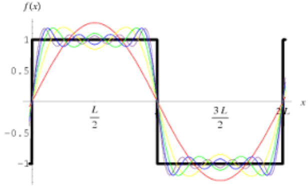
- 

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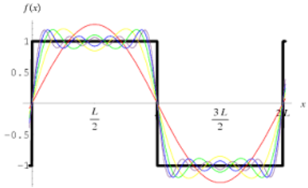
- 

- **"Fourier's theorem has all the simplicity and yet more power than other familiar explanations in science. Stated simply, any complex patterns, whether in time or space, can be described as a series of overlapping sine waves of multiple frequencies and various amplitudes** - Bruce Hood (clinical psychologist)

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$$\{e^{2\pi i x\cdot\lambda}\}_{\lambda\in\Lambda},$$

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where Λ is a discrete set that shall be referred to as a **spectrum**.

- **BASIC POINT CONFIGURATION QUESTION:** Given a discrete set of points with a suitable density condition, this this set contain a congruent or similar copy of a given point configuration?

Orthogonal exponential bases

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- Fuglede proved that this conjecture holds if either the **tiling set** for Ω or the **spectrum** (that generates the orthogonal exponential basis) is a **lattice**.
- The **Fuglede Conjecture** was disproved by Terry Tao in 2003, yet it holds in many cases and continues to inspire compelling research combining combinatorial, arithmetic and analytic techniques.

Orthogonal exponential bases on the ball

- Let $\Omega = [0,1]^d$. Then $\{e^{2\pi i x \cdot k}\}_{k \in \mathbb{Z}^d}$ is an orthonormal basis for $L^2(\Omega)$, the classical Fourier series.

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- If $\{e^{2\pi i x \cdot \lambda}\}_{\lambda \in \Lambda}$ is an orthonormal basis for $L^2(B_d)$, then Λ is separated and has density $|B_d|$ by the classical Beurling density theorem.
- Orthogonality means that for every $\lambda \neq \lambda' \in \Lambda$,

$$\widehat{\chi}_{B_d}(\lambda - \lambda') = \int_{B_d} e^{2\pi i x \cdot (\lambda - \lambda')} dx = 0.$$

Implications of orthogonality

- Orthogonality implies that for any $\lambda \neq \lambda' \in \Lambda$,

$$\widehat{\chi}_{B_d}(\lambda - \lambda') = 2\pi|\lambda - \lambda'|^{-\frac{d}{2}} J_{\frac{d}{2}}(2\pi|\lambda - \lambda'|) = 0,$$

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- Since zeroes of $J_{\frac{d}{2}}$ are uniformly separated,

$$\#\{|\lambda - \lambda'| : \lambda, \lambda' \in \Lambda \cap [-R, R]^d\} \leq CR,$$

- while the density of Λ implies that

$$\#\left\{\Lambda \cap [-R, R]^d\right\} \approx R^d,$$

and this transforms the question about orthogonal exponential bases into one about distance sets of point configurations.

The Erdős Distance Problem

Conjecture

(Erdős, 1945) The set of size R^d *in* $\mathbb{R}^d$, $d \geq 2$, *determines* $\gtrapprox R^2$ *distinct distances.*

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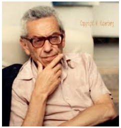

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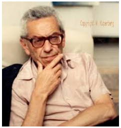
- 
- This is only known in $\mathbb{R}^2$ (Guth-Katz Ann. of Math. 2015), but the fact that the number of distinct distances is $\geq CR^\alpha$ for some $\alpha > 1$ was established back in 1953 by Leo Moser.

Establishing a contradiction

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- We have shown that the size of P is $\approx R^d$, while P determines $\leq CR$ distinct distances.
- This is impossible by the aforementioned results on the Erdős distance problem.
- It follows that $L^2(B_d)$ does not possess an orthogonal basis of exponentials (Iosevich-Katz-Pedersen, MRL (1999)).

The Erdős Integer Distance Principle

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- **The Erdős Integer Distance Principle:** Let $A \subset \mathbb{R}^d$ be an infinite set such that $\Delta(A) \equiv \{|x-y| : x, y \in A\} \subset \mathbb{Z}$. Then $A \subset$ line.

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Theorem

(A.I. and M. Rudnev, IMRN (2003)) Let K be a bounded convex symmetric body with a smooth boundary and everywhere non-vanishing Gaussian curvature and let $\{e^{2\pi i x \cdot a}\}_{a \in A}$ denote a set of orthogonal exponentials in $L^2(K)$. If $d \neq 1 \mod 4$, then A is finite. If $d = 1 \mod 4$, A may be infinite. If A is infinite, it is a subset of a line.

The Erdős Integer Distance Principle-the point

- If K is a symmetric with a C^∞ boundary and non-vanishing curvature, then $\widehat{\chi}_K(\xi)$ is equal to

$$C\kappa^{-\frac{1}{2}}\left(\frac{\xi}{|\xi|}\right)\sin\left(2\pi\left(\rho^*(\xi)-\frac{d-1}{8}\right)\right)|\xi|^{-\frac{d+1}{2}}+O(|\xi|^{-\frac{d+3}{2}}),$$

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- where κ is the Gaussian curvature at the point on ∂K where $\frac{\xi}{|\xi|}$ is the unit normal, $K=\{x:\rho(x)=1\}$ and $\rho^*(\xi)=\sup_{x\in\partial K} x\cdot\xi$.

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- From this formula we deduce that if $e^{2\pi i x\cdot a}$ and $e^{2\pi i x\cdot a'}$ are orthogonal in $L^2(K)$, then $\rho^*(a-a')$ is, up to a small error, a shifted integer.

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- It turns out that the Erdős Integer Distance Principle still holds in this setting, with the Euclidean norm replaced by a more general (smooth) norm, and integer distances replaced by shifted integers up to a small asymptotic error.

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- The result above raises the question of just of just how large can a set of exponentials orthogonal on $L^2(B_d)$ possibly be?
- It is quite probable that the answer is $d+1$, though the only thing resembling a quantitative result in this direction is a theorem due to Iosevich and Kolountzakis (APDE 2013) which says that if $\{e^{2\pi ix\cdot a}\}_{a\in A}$ is orthogonal on $L^2(D)$, D the unit disk in $\mathbb{R}^2$, then $\#\{A\cap[-R,R]^2\}\leq CR^{\frac{2}{3}}$.

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(Furstenberg, Katznelson and Weiss (1986)) Let $E \subset \mathbb{R}^d$ *be a set of positive upper Lebesgue density, in the sense that* $\limsup_{R \to \infty} \frac{|E \cap B(x,r)|}{|B(x,r)|} = c > 0$. *Then there exists a threshold* $l(E)$ *such that for all* $l' > l$, *there exist* $x, y \in E$ *such that* $|x-y| = l'$. *In other words, every sufficiently large distance is realized.*

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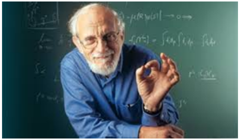

Applying the Furstenberg-Katznelson-Weiss result

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Applying the Furstenberg-Katznelson-Weiss result

- If $\{e^{2\pi i x\cdot\lambda}\}_{\lambda\in\Lambda}$ is an orthonormal basis for $L^2(B_d)$, then Λ is separated and has density $|B_d|$ by the classical Beurling density theorem.

- Thicken each point of Λ by a small $\delta > 0$. The resulting set has positive upper Lebesgue density, so Furstenberg-Katznelson-Weiss implies that every sufficiently large distance is realized.

  By orthogonality,

$$0 = \widehat{\chi}_{B_d}(\lambda - \lambda) = |\lambda - \lambda'|^{-\frac{d}{2}} J_{\frac{d}{2}}(2\pi|\lambda - \lambda'|).$$

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- Since zeroes of $J_{\frac{d}{2}}$ are uniformly separated, it is impossible to recover every sufficiently large distance and we have a contradiction.

A stronger formulation

Definition

We say that $\mathcal{E}(\Lambda) = \{e^{2\pi i x \cdot \lambda}\}_{\lambda \in \Lambda}$ is a ϕ-approximate orthogonal basis for $L^2(\Omega)$, Ω a bounded domain in $\mathbb{R}^d$, if $\mathcal{E}(\Lambda)$ is a basis and

$$|\widehat{\chi}_\Omega(\lambda - \lambda')| \leq \phi(|\lambda - \lambda'|),$$

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Theorem

(A. Iosevich and A. Mayeli (2020)) Let ϕ be a any function such that

$$\lim_{t \to \infty} (1 + t)^{\frac{d+1}{2}} \phi(t) = 0.$$

Then there does not exist a set Λ such that $L^2(B_d)$ possesses a ϕ-approximate orthogonal basis $\mathcal{E}(\Lambda)$.

A sketch of the proof

- Applying Furstenberg-Katznelson-Weiss to the δ-nerighborhood of Λ, denoted by E_δ, it follows that there exists $L_0 > 0$ such that for every $L > L_0$, there exist $x, x' \in E_\delta$ with

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- Applying the assumption and the asymptotic formula for $\widehat{\chi}_{B_d}$, we see that for given $\epsilon > 0$ there exists $R > 0$ such that if $\lambda, \lambda' \in \Lambda$ with $|\lambda - \lambda'| > R$, then

$$\left| \sin\left(2\pi\left(|\lambda - \lambda'| - \frac{d-1}{8}\right)\right)\right| < \epsilon.$$

Thickened Λ

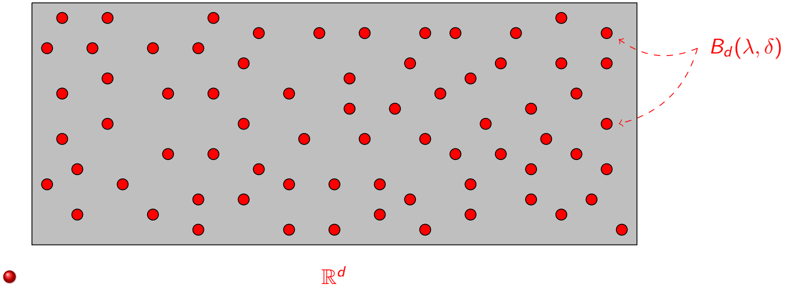

Figure: In this graph, each node represents a ball centered at $\lambda \in \Lambda$ with radius δ.

A sketch of the proof (continued)

- It follows that for some positive integer k we must have

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- It follows that for every $\epsilon > 0$ there exists $R > 0$ such that if $|x - x'| > R$, $x, x' \in E_\delta$, then

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- If ϵ and δ are taken to be sufficiently small, then any sufficient large distance L can not be realized in the set E_δ. This contradiction establishes the theorem.

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- But what if such an "approximate orthogonality" condition only holds on average?

Further connections: Bourgain enters the picture

- As we dig deeper, more and more configuration scenarios come up!
- The previous result we discussed involved "approximate orthogonality" in the sense that orthogonality held up to a small point-wise error.
- But what if such an "approximate orthogonality" condition only holds on average?
- As we shall see in a moment, a convenient tool to address such situation is a pinned variant of the Furstenberg-Katznelson-Weiss result established by Bourgain.

Average orthogonality

Theorem

Let A be a separated subset of $\mathbb{R}^d$, $d \geq 2$. Let $p \in [1, \infty]$. Let K denote a bounded symmetric convex set with a smooth boundary and everywhere non-vanishing Gaussian curvature. Let $\phi \geq 0$ be a continuous monotonically non-increasing function on $[0, \infty)$ such that there exists a sequence $\{R_j\} \to \infty$, so that for any $\epsilon > 0$,

$$\left(\frac{1}{R_j} \int_{R_j}^{2R_j} \phi^p(t)dt \right)^{\frac{1}{p}} \leq \epsilon \left(R_j^{-\frac{d+1}{2}} \right)$$

if j is sufficiently large, and suppose that $|\widehat{\chi}(a - a')| \leq \phi(\rho^*(a - a'))$ for all $a \neq a$, $a, a' \in A$, where ρ^* is the Minkowski functional on K^*, the dual body of K. Then the upper density of A is equal to 0. Consequently, $\{e^{2\pi i x \cdot a}\}_{a \in A}$ **is not** *a frame for $L^2(K)$.*

Average orthogonality-technical formulation

Theorem

Let A be a separated subset of $\mathbb{R}^d$, $d \geq 2$. Let K denote a bounded symmetric convex set with a smooth boundary and everywhere non-vanishing Gaussian curvature. Let B_{R_j}, B'_{R_j} denote balls of radius R_j with centers separated by $3R_j$, where R_j is a sequence of positive real numbers tending to infinity. There exists $c > 0$ such that if for any

$$\left(\frac{1}{R_j^{2d}} \sum_{a \in A \cap B_{R_j}, a' \in B'_{R_j}} |\widehat{\chi}_K(a - a')|^p \right)^{\frac{1}{p}} \leq cR_j^{-\frac{d+1}{2}},$$

then the upper density of A is equal to 0.

Sketch of proof

- By the method of stationary phase

$$|\widehat{\chi}_K(a-a')| \leq C(1+|a-a'|)^{-\frac{d+1}{2}}$$

for some constant $C > 0$. It follows that the assumption of the theorem always holds with **some constant**.

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$$|\widehat{\chi}_K(a-a')| \leq C(1+|a-a'|)^{-\frac{d+1}{2}}$$

for some constant $C > 0$. It follows that the assumption of the theorem always holds with **some constant**.

- The point is that if it holds with a small enough constant, we can invoke the asymptotic formula and argue that

$$\sin\left(2\pi\left(\rho^*(a-a') + \frac{d-1}{8}\right)\right)$$

is small **quite often** for $a \neq a'$, $a, a' \in A$.

Pinned point configurations to the rescue

- By pigeonholing, we will be able to conclude that there exists a point x_0 living in a small neighborhood of an $a_0 \in A$, such that pinned ρ^*-distances from x_0 to the small neighborhood of a positive proportion of A is clustered around

$$\frac{k}{2} + \frac{d-1}{8},$$

where k ranges over the integers.

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- This violates a variant of the Furstenberg-Katznelson-Weiss result due to Bourgain which says that in sets of positive upper Lebesgue density, a pinned distance set with respect to almost every point contains every sufficiently large positive real number.