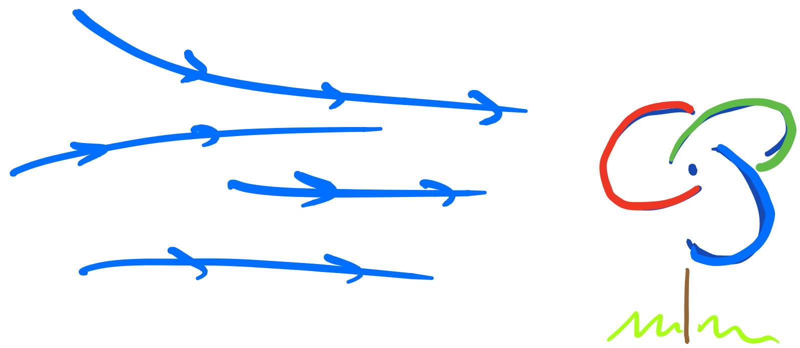
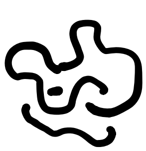
Measuring Chirality with the Wind



2021

 Surfaces

 *curves in $\mathbb{R}^3$

 bodies

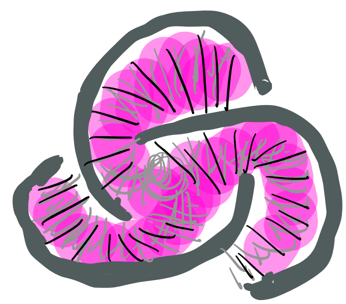
1

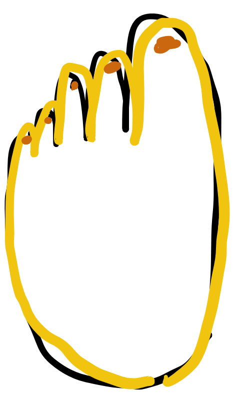
I call (______) Chiral if its image in a plane mirror cannot be brought to coincide with itself.

— Lord Kelvin

Left Right

Left





Right

Is there a (cts) scalar measure of chirality that distinguishes Handedness ?

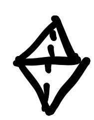
$$\chi(\cdot) \begin{cases} > 0 & \text{Right-handed} \\ < 0 & \text{Left-handed} \\ = 0 & \text{Achiral} \end{cases}$$

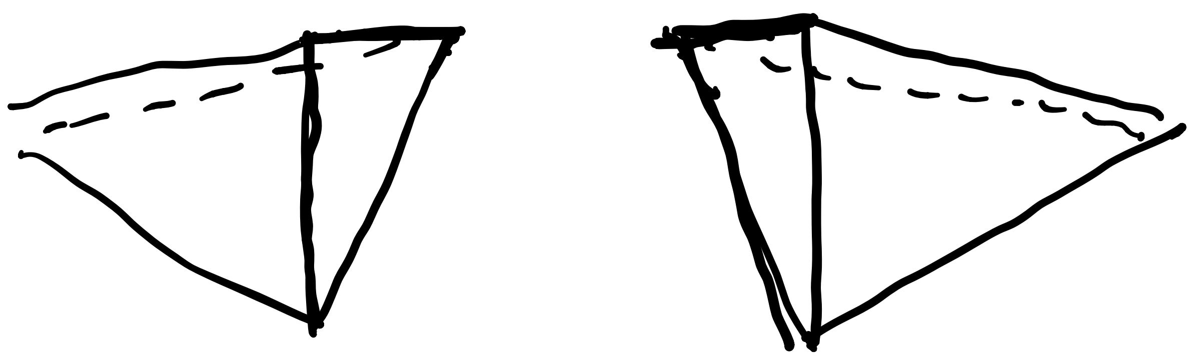
Deformations and Chiral connectedness.

Sadly, no such scalar measure exists. (in a reasonable way....)

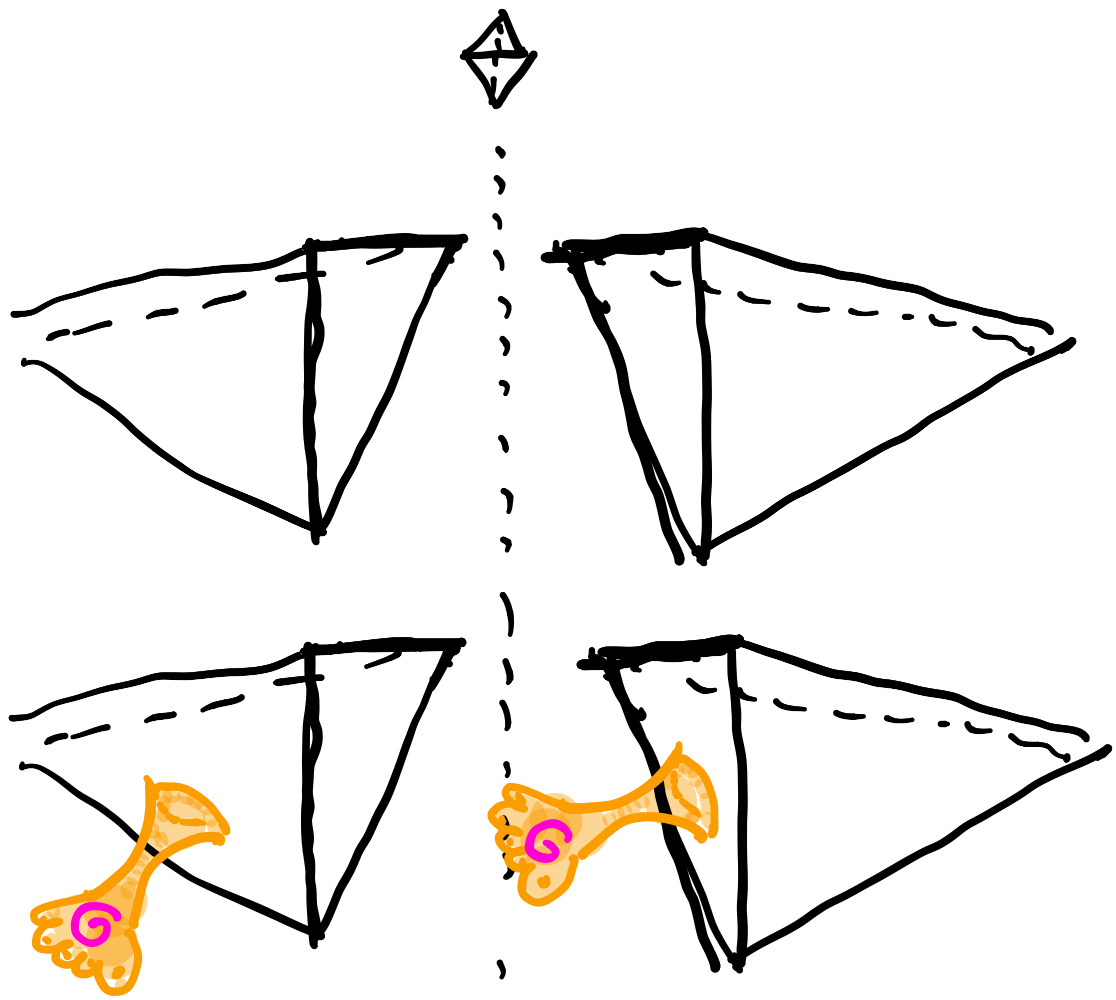
There are paths from "L" to "R" through purely chiral configurations

For Curves & Surfaces of sufficient regularity:





For Curves & Surfaces of
└ Sufficient regularity:



②

Hydrodynamic Intuition

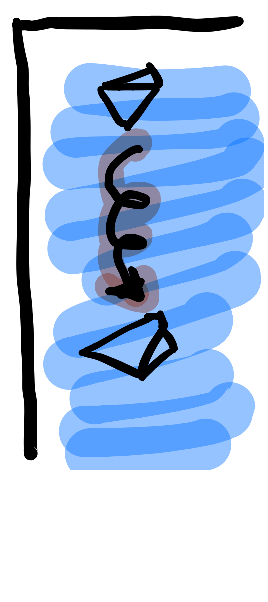
also back → to Kelvin:

Things twist in the wind.

- Viscous/inviscid flow has an effect.

L+L, H+B

Kelvin, Fawcett:



Equations of motion of a body in a fluid.

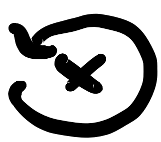
not so fun :(

But here's a

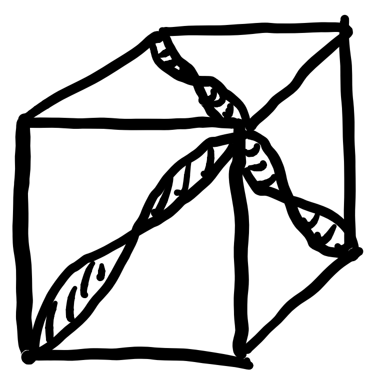
Motivational Question:

Is there an Isotropic Helicoid?

counterclockwise



clockwise



???

(DRKKS)

Turnabout: Use this to study shape using fluid as a measurement tool.

Approximation with 1st order scattering, Infinitesimal motion.

Again: Things twist in the wind.

3 Curves in $\mathbb{R}^3$

force at x is proportional to

$$f_v = v - t(t\cdot v)$$

$$= \underbrace{(I - tt^*)}_{\text{force density operator}}\vec{v}$$

force density operator: projection of v to normal plane at x.

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analogous projection to normal for surfaces

This force can be applied to the lever arm $x(s)$ yielding torque measured at O

$$q = x \times f.$$

$[x \times \_]$ linear, represented by

$$\begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

Observe:

$X_{x\_} \in \mathrm{Skew}(M)$

$f \in \mathrm{Sym}(M)$

$\Rightarrow \mathrm{Tr}[q] = 0$

Also, we may integrate f and q on curve

$\rightsquigarrow$ F and Q,

Symmetric and Tracefree respectively.

(5)

Torque matrix / operator

for $u \in S^2$ direction

$u^* Q v$ is the component of torque about axis defined by u thru $\vec{O}$.

$\underline{u^* Q u}$ measure "twist" of curve about the axis parallel to the wind.

Remark:

This is a Quadratic Form.

So Sym Q is the only contributor to this twist, still trace free.

Also, f and q (and F and Q) satisfy transformation laws

- "$\vec{a} + f$" $= f$
- "Rf" $= RfR^*$
- "$\vec{a} + q$" $= q + \vec{a} \times f$
- "Rq" $= \left(\begin{smallmatrix}\text{sign} \\ \det R\end{smallmatrix}\right) RqR^*$

Can find interesting

<u>Centers</u> :

- Center of mass

- Minimize various norms

- Q symmetric
  (center of reaction)

Ex. <u>Solve for this</u>

$$\mathrm{Skew}(\bar{a} \times F) = \mathrm{Skew}(Q)$$
$$= 0 \;\ldots$$

- $F = \Delta F_i$, $\bar{a} = \left\{ \dfrac{Q_{ji} - Q_{ji}}{F_i + F_j} \right\}$

⑥ Implications for Kelvin's helicoid.

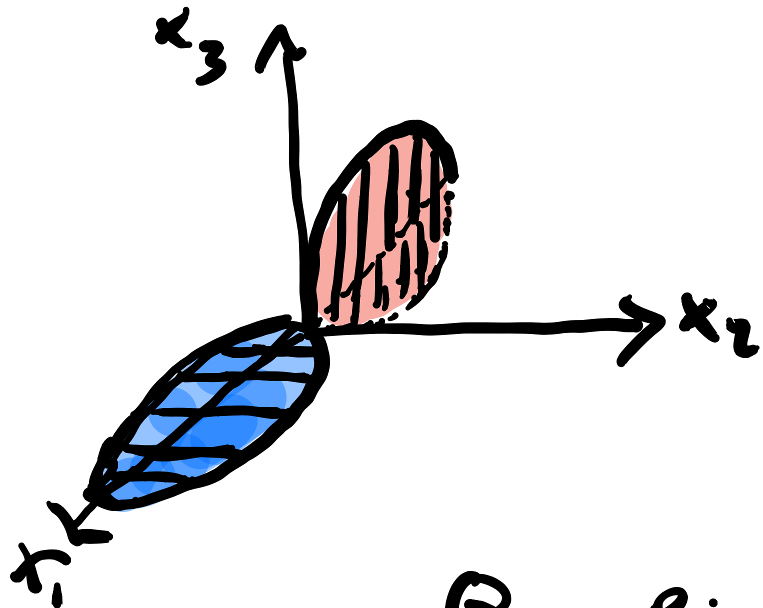
If it twists to the left, it also twists to the right.

In fact, the "average" twist is always 0.

Also: we can compute

Q

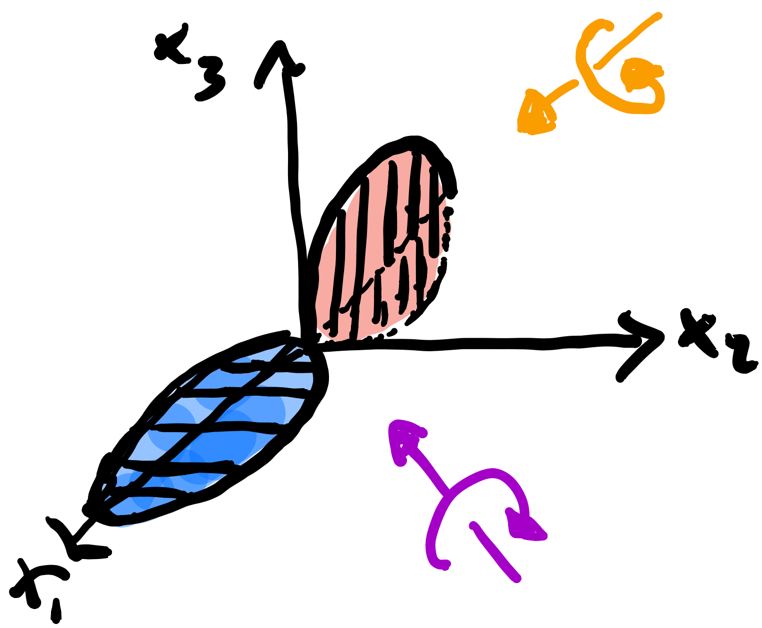
Ex.



$$\begin{array}{lcc} & Q & e_i \\ \vec{0} \propto & \begin{bmatrix}0 & & \\ 0 & & \\ 0 & & \end{bmatrix} & \begin{bmatrix}1\\0\\0\end{bmatrix} \\ -e_3 \propto & \begin{bmatrix} & 0 & \\ & 0 & \\ & -1 & \end{bmatrix} & \begin{bmatrix}0\\1\\0\end{bmatrix} \\ -e_2 \propto & \begin{bmatrix} & & 0 \\ & & -1 \\ & & 0\end{bmatrix} & \begin{bmatrix}0\\0\\1\end{bmatrix} \end{array}$$

- $u_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ $\quad u_1 Q u_1 = -2$

- $u_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ $\quad u_2 Q u_2 = 2$



Can realize any Q using this type of construction

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How "handed" is this?

If Q is our tensorial measure of chirality

It is both left- and right handed in equal total amount.

But also: $x_1 = -2x_2 = -2x_3$