Discretization of integrals on compact metric measure spaces

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March 10, 2021

This paper is a joint work with Professor Martin D. Buhmann and Professor Feng Dai

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- A prominent example is seeking effective ways of approximating an integral

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via the weighted summation

$$\Lambda_N(\Phi(\rho(x,\cdot),\xi) = \sum_{j=1}^{N} \lambda_j \Phi(\rho(x,y_j)),$$

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via the weighted summation

$$\Lambda_N(\Phi(\rho(x,\cdot),\boldsymbol{\xi}) = \sum_{j=1}^{N} \lambda_j \Phi(\rho(x,y_j)),$$

where (X,ρ) is a compact metric space, they can be considered as a *discretization* of probability measures on X.

Motivation

- This formula $\Lambda_N(\Phi(\rho(x,\cdot)),\boldsymbol{\xi})$ is called a *cubature formula* (C.F.) with nodes $\boldsymbol{\xi} = (y_1, \ldots, y_N) \in X^N$ and weights $\Lambda := (\lambda_1, \ldots, \lambda_N) \in \mathbb{R}^N$, it is called positive if $\lambda_1, \cdots, \lambda_N \geq 0$

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- The error of such approximation is measured by the following quantity:

$$\Lambda_N(\mathbb{W},\boldsymbol{\xi}) := \sup_{\Phi(\rho(x,\cdot))\in\mathbb{W}} \left| \int_X \Phi(\rho(x,y))g(y)\,\mathrm{d}\mu(y) - \Lambda_N(\Phi(\rho(x,\cdot),\boldsymbol{\xi}) \right|$$

One can further optimize the C.F.s and study the quantity $\inf_{\Lambda_N} \Lambda_N(\mathbb{W},\boldsymbol{\xi})$

1. Using the Bernstein inequality in probability, one can show that if $f \in C(\mathbb{S}^d)$, and $\xi_1, \cdots, \xi_N$ are independent random points selected uniformly on the sphere $\mathbb{S}^d$, then there exists an absolute constant $c_1 > 0$ such that the inequality

$$\Big| \int_{\mathbb{S}^d} f \, d\mu_d - \frac{1}{N} \sum_{j=1}^{N} f(\xi^j) \Big| \leq t N^{-\frac{1}{2}}, \quad t > 1$$

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2. The proof of main results in this paper mainly follows along the same idea as that of [1].

Main results

Introduction

Let (Ω, ρ) be a compact metric space.

(a) Open balls and closed balls in Ω will be denoted by $B_\zeta(x) := \{y \in \Omega : \rho(x, y) < \zeta\}$, and $B_\zeta[x] := \{y \in \Omega : \rho(x, y) \leqslant \zeta\}$, respectively.

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(d) Given a set $A \subset \Omega$ and a point $x \in \Omega$, define

$$\mathrm{dist}(x, A) := \inf_{y \in A} \rho(x, y).$$

Regular partitions on compact metric space

Regular partitions on compact metric space

(a) A measure μ on Ω is called non-atomic if for any measurable set $A \subset \Omega$ with $\mu(A) > 0$ there exists a measurable subset B of A such that $\mu(A) > \mu(B) > 0$.

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(a) A measure μ on Ω is called non-atomic if for any measurable set $A \subset \Omega$ with $\mu(A) > 0$ there exists a measurable subset B of A such that $\mu(A) > \mu(B) > 0$.

(b) For non-atomic Borel probability measure μ on Ω, we have the property: If $A_0 \subset A_1 \subset \Omega$, $0 < \mu(A_1)$ and $\mu(A_0) \leq t \leq \mu(A_1)$, then there exists a measurable subset $E_t \subset A_1$ satisfies $\mu(E_t) = t$.

Regular partitions on compact metric space

Theorem (2.1)

Let (Ω, ρ) *be a compact path-connected metric space with diameter diam*$(\Omega) := \max_{x,y\in\Omega} \rho(x,y) = \pi$. *Let* μ *be a non-atomic Borel probability measure on* Ω, *and* $N \geq 2$ *a positive integer. Assume that the inequality*

$$\inf_{x\in\Omega} \mu\Big(B_{\delta/2}(x)\Big) \geqslant \frac{1}{N} \tag{1}$$

holds for some $\delta > 0$. *Then there exists a partition* $\{R_1, \ldots, R_N\}$ *of* Ω *such that*

1. *the* R_j *are pairwise disjoint subsets of* Ω,
2. *for each* $1 \leq j \leq N$, $\mu(R_j) = \frac{1}{N}$ *and* $\mathrm{diam}(R_j) \leq 4\delta$.

Regular partitions on compact metric space

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Regular partitions on compact metric space

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2. The crucial point here lies in the fact that the constant 4 in the estimates of $\mathrm{diam}(R_j)$ is absolute.

Discretization on compact metric spaces

Discretization on compact metric spaces

Let (X,ρ) be a compact metric space with metric ρ and diameter π. For $x \in X$ and $0 \le a < b \le \pi$, set

$$E(x; a, b) := \{y \in X : \quad a \le \rho(x, y) \le b\}.$$

A partition of $[a, b]$ consists of finitely many pairwise disjoint subsets.

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Definition

Let $0 = t_0 < t_1 < \cdots < t_\ell = \pi$ *be a partition of the interval* $[0, \pi]$, *and let* $r \in \mathbb{N}$. *We say* $\Phi \in C[0, \pi]$ *belongs to the class* $\mathcal{S}_r \equiv \mathcal{S}_r(t_1, \ldots, t_\ell)$ *if there exists an* r*-dimensional linear subspace* V_r *of* $C(X)$ *such that for any* $x \in X$ *and each* $1 \le j \le \ell$,

$$\Phi(\rho(x, \cdot))\Big|_{E(x; t_{j-1}, t_j)} \in \left\{ f\Big|_{E(x; t_{j-1}, t_j)} : \quad f \in V_r \right\}.$$

Discretization on compact metric spaces

Let μ be a Borel probability measure on X satisfying the following condition for a parameter $\beta \geq 1$ and some constant $c_1 > 1$:

Condition (a)

For each positive integer N, there exists a partition $\{X_1, \ldots, X_N\}$ of X such that $\mu(X_j) = \frac{1}{N}$ and $\text{diam}(X_j) \leq \delta_N := c_1 N^{-\frac{1}{\beta}}$ for $1 \leq j \leq N$.

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According to Theorem 2.1, this condition holds automatically with $c_1 = 20\pi$ if: the metric space X is path-connected, and μ is a non-atomic Borel probability measure on X satisfying that for any $0 < t \leq 1$,

$$\inf_{x \in X} \mu(B_t(x)) \geq \left(\frac{8}{c_1}\right)^{\beta} t^{\beta}. \tag{2}$$

Discretization on compact metric spaces

Let $\Phi \in C[0,\pi]$ satisfy

$$|\Phi(s) - \Phi(s')| \le |s - s'|, \qquad \forall s, s' \in [0,\pi], \tag{3}$$

and belong to a class $\mathcal{S}_r(t_1, \ldots, t_\ell)$ for some compact metric space (X, ρ), where $r \in \mathbb{N}$ and $0 = t_0 < t_1 < \cdots < t_\ell = \pi$.

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Condition (b)

For each $x \in X$ and $\delta \in (0, \pi)$,

$$\mu\Big(E(x; t_j - \delta, t_j + \delta)\Big) \leqslant c_2\delta, \;\; 1 \leq j < \ell, \tag{4}$$

where $c_2 > 1$ is a constant independent of δ and x.

Discretization on compact metric spaces

Theorem (2.3)

Let $\Phi \in C[0,\pi]$ *and a Borel probability measure* μ *on* X *satisfying the above conditions, then for each positive integer* $N \geq 4$*, there exist points* $y_1, \ldots, y_{(r+2)N} \in X$ *and nonnegative numbers* $\lambda_1, \ldots, \lambda_{(r+2)N} \geqslant 0$ *such that* $\sum_{j=1}^{(r+2)N} \lambda_j = 1$ *and*

$$\max_{x\in X} \left| \int_X \Phi(\rho(x,y))\, \mathrm{d}\mu(y) - \sum_{j=1}^{(r+2)N} \lambda_j \Phi(\rho(x,y_j)) \right| \leqslant c_3 N^{-\frac{1}{2}-\frac{3}{2\beta}} \sqrt{\log N},$$

where $c_3 := 8c_1^2\sqrt{c_2\ell}\sqrt{\beta}$.

Discretization on compact metric spaces

Theorem (2.4)

Let (X, ρ) be a compact path-connected metric space. Let $\Phi \in C[0, \pi]$ satisfy (3) and belong to a class $\mathcal{S}_r(t_1, \ldots, t_\ell)$ for some $r \in \mathbb{N}$ and $0 = t_0 < t_1 < \cdots < t_\ell = \pi$. Let μ be a non-atomic Borel probability measure on X satisfying (2). Assume in addition that the Condition (b) in Theorem 2.3 is satisfied.

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$$\max_{x \in X} \left| \int_X \Phi(\rho(x, y)) g(y) \, \mathrm{d}\mu(y) - \sum_{j=1}^{2(r+2)N} \lambda_j \Phi(\rho(x, y_j)) \right|$$

$$\leqslant 45 c_3 N^{-\frac{1}{2} - \frac{3}{2\beta}} \sqrt{\log N}.$$

Discretization on finite-dimensional compact domains

Introduction

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(ii) Let $\Omega \subset B_1[0]$ be a compact subset of X (not necessarily connected).

(iii) μ be a Borel probability measure supported on Ω.

Discretization on finite-dimensional compact domains

We assume that the probability measure μ satisfies the following two conditions:

1. there exist a positive constant $c_4 > 1$ and a parameter $\beta \geq 1$ such that for any $x \in \Omega$ and $\delta \in (0, 2]$

$$c_4^{-1}\delta^{\beta} \leq \mu\Big(B_{\delta}(x)\Big) \leq c_4\delta^{\beta}; \tag{5}$$

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(ii) there exists a constant $c_5 > 0$ such that for any $x \in \Omega$ and $t, s \in (0, 2]$,

$$\mu\Big(\{y \in \Omega: \; t \leq \|y - x\| \leq t + s\}\Big) \leq c_5 s. \tag{6}$$

Discretization on finite-dimensional compact domains

Let $\Phi : [0, \infty) \to \mathbb{R}$ be a function such that

$$|\Phi(s) - \Phi(s')| \leq |s - s'|, \ \forall s, s' \in [0, 2]. \tag{7}$$

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Assume that there exist a partition $0 = t_0 < t_1 < \cdots < t_\ell = 2$ of $[0, 2]$ and a translation-invariant linear subspace X_r of $C(\Omega)$ with $\dim X_r = r$ such that with $E_j := \{x \in X : \quad t_{j-1} \leq \|x\| \leq t_j\}$, $j = 1, 2, \dots, \ell$,

$$\Phi(\|\cdot\|)\Big|_{E_j} \in \Big\{ f\Big|_{E_j} : \quad f \in X_r \Big\}.$$

Let $g \in L^1(\Omega, \mu)$ be such that $\|g\|_{L^1(d\mu)} = 1$.

Discretization on finite-dimensional compact domains

Under these two conditions, we prove

Theorem

For each positive integer $n \geq 2$, there exist points $y_1, \ldots, y_n \in \Omega$ and real numbers $\lambda_1, \ldots, \lambda_n$, such that

$$\sup_{x \in \Omega} \left| \int_\Omega \Phi(\|x-y\|) g(y) \, \mathrm{d}\mu(y) - \sum_{k=1}^{n} \lambda_k \Phi(\|x-y_k\|) \right|$$

$$\leq C(X) \begin{cases} n^{-\frac{1}{2}-\frac{3}{2\beta}} (\log n)^{\frac{1}{2}}, & \text{if } 1 \leq \beta < 3, \\ n^{-1} (\log n)^{\frac{3}{2}}, & \text{if } \beta = 3, \\ n^{-\frac{\beta+1}{2(\beta-1)}} (\log n)^{\frac{1}{2}}, & \text{if } \beta > 3, \end{cases} \tag{8}$$

where the constant $C(X)$ depends only on $\dim X$, c_4, c_5, r, ℓ and β.

Example: Discretization on the unit sphere $\mathbb{S}^d$

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When $X = \mathbb{S}^d \subset \mathbb{R}^{d+1}$ be the unit sphere of $\mathbb{R}^{d+1}$ equipped with the normalized surface Lebesgue measure μ_d and the geodesic distance $\rho(x, y) = \arccos(x \cdot y), x, y \in \mathbb{S}^d$. We have:

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Lemma (3.1)

For any positive integer N,

$$\inf_{x \in \mathbb{S}^d} \mu_d(B_{\delta_N}(x)) \geq \frac{1}{N} \quad \textit{with} \quad \delta_N := 5\pi N^{-\frac{1}{d}}. \tag{9}$$

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Lemma

For any $\delta > 0$, $x \in \mathbb{S}^d$ and $t \in (0, \pi)$,

$$\mu_d\left(\left\{y \in \mathbb{S}^d : \ t - \delta \leqslant \rho(x, y) \leqslant t + \delta\right\}\right) \leqslant \frac{3}{2}\sqrt{d}\delta. \tag{10}$$

Example: Discretization on the unit sphere $\mathbb{S}^d$

As a consequence of Theorem 2.1 and Lemma 3.1, we have that

Theorem

For each integer $N \geq 1$*, there exists a partition* $\{R_1, \ldots, R_N\}$ *of* $\mathbb{S}^d$ *such that*

1. *the* R_j *are pairwise disjoint subsets of* $\mathbb{S}^d$*;*
2. *for each* $1 \leq j \leq N$*,* $\mu_d(R_j) = \frac{1}{N}$ *and* $diam(R_j) \leq 40\pi N^{-\frac{1}{d}}$*.*

Example: Discretization on the unit sphere $\mathbb{S}^d$

Again, the main point here is that the upper bound for $N^{\frac{1}{d}} \max_j \operatorname{diam}(R_j)$ is independent of the dimension d. Theorem 3.3 with dimension dependant upper bound for $N^{\frac{1}{d}} \max_j \operatorname{diam}(R_j)$ can be found in [2].

Example: Discretization on the unit sphere $\mathbb{S}^d$

Theorem (3.4)

Let $\Phi : [-1,1] \to \mathbb{R}$ be a piecewise polynomial of degree at most r with knots $-1 = s_0 < s_1 < \cdots < s_\ell = 1$ such that $|\Phi(s) - \Phi(s')| \le |s - s'|$ for any $s, s' \in [-1,1]$. Let $m_r = m_r^d$ denote the dimension of the space of all spherical polynomials of degree at most r on $\mathbb{S}^d$. Let $g \in L^\infty(\mathbb{S}^d)$ be such that $\|g\|_\infty \le 1$. Then for each positive integer $N \ge 20$, there exist points $\xi_1, \ldots, \xi_{2(m_r+2)N} \in \mathbb{S}^d$ and real numbers $\lambda_1, \ldots, \lambda_{2(m_r+2)N}$ such that

$$\max_{x \in \mathbb{S}^d} \left| \int_{\mathbb{S}^d} \Phi(x \cdot y) g(y) \, \mathrm{d}\mu_d(y) - \sum_{j=1}^{2(m_r+2)N} \lambda_j \Phi(x \cdot \xi_j) \right|$$
$$\le 7 \cdot 10^6 \sqrt{\ell} \cdot d^{\frac{3}{4}} N^{-\frac{1}{2} - \frac{3}{2d}} \sqrt{\log N}.$$

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THANK YOU!