DIMENSIONAL ANALYSIS & SCALING

- The numerical value of any quantity in a mathematical model is measured with respect to a system of units.

For example: meters in a mechanical model
dollars in a financial model

- The units used to measure a quantity are arbitrary and a change in the system of units cannot change the model.

For example: $v = \frac{d}{t}$ average speed $= \frac{\text{distance}}{\text{time}}$

it doesn't matter if the distance is measured in meters or feet (or anything else) and/or time is measured in hours or seconds (etc...)

- The value of a dimensional quantity may be measured as some multiple of a basic unit. We already do this every day, but the basic unit is usually arbitrary.

For example: 1 inch = width of an average man's thumb at the base of the nail

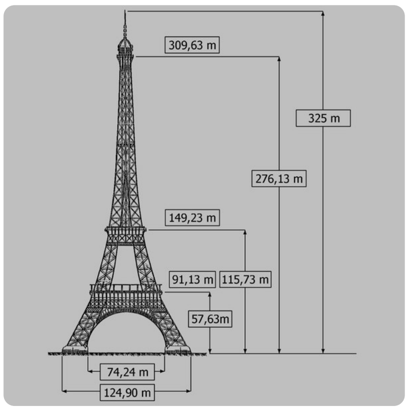
according to King David I of Scotland in 1150

* a change in the system of units leads to a rescaling of the quantities it measures

* the ratio of two quantities with the same unit does not depend on the particular choice of the system of units

$\hookrightarrow$ SCALE-INVARIANCE of the model

For example: say I want to measure the height of the Eiffel tower in Paris. I can measure it in :



- meters $\longrightarrow$ 325 m
- feet $\longrightarrow$ 1,063 ft

⋮

- baguettes $\longrightarrow$ ~ 500

  1 baguette ~ 65 cm (26 in) long

- A FUNDAMENTAL SYSTEM OF UNITS (FSU) is a set of independent units from which all other units in the system can be derived.

  * the choice of FSU for a certain class of problems is not unique

  * given a FSU, any other unit in the system may be constructed uniquely as a product of powers of the fundamental units

For example: for mechanical problems a suitable FSU can be:

- mass, length, time $\longrightarrow$ velocity $\quad V = L\,T^{-1}$

  M L T $\longrightarrow$ force $\quad F = M L T^{-2}$

  OR pressure $\quad P = M L^{-1} T^{-2}$

- force, length, time $\longrightarrow$ mass $M = FL^{-1}T^{-2}$
  F L T $\longrightarrow$ velocity $V = LT^{-1}$
  pressure $P = FL^{-2}$

NOTE

- Distinction between fundamental and derived unit is arbitrary
- Number of fundamental units is arbitrary

↳ We may reduce the number of fundamental units by taking advantage of the dimensional constants in the system

For example: in relativistic mechanics, if we use M, L, T as our FSU, then the speed of light c is defined as:

$$c = 3 \times 10^8 \ m s^{-1} \longleftarrow \text{dimensional constant}$$

$$[c] = LT^{-1}$$

We can choose to set $c = 1$ and use M, T as FSU

↳ then we measure lengths in travel-time of light $\longrightarrow$ the light-year is the distance that light travels in vacuum in one year:

$$1 \ ly = 9.4607 \times 10^{15} \ m$$

$$= 5.8786 \times 10^{12} \text{ miles}$$

SCALING and NON-DIMENSIONALIZATION

- Let $(d_1, d_2, \ldots, d_r)$ denote a FSU and a denote a quantity that's measurable with respect to this system. Then the dimension of a, written $[a]$, is given by

$$[a] = d_1^{\alpha_1} d_2^{\alpha_2} \ldots d_r^{\alpha_r} \qquad \text{for suitable exponents } (\alpha_1, \alpha_2, \ldots, \alpha_r)$$

For example: back to mechanics and $(d_1, d_2, d_3) = (M, L, T)$

$\hookrightarrow$ $[F] = M^1 L^1 T^{-2}$ $\qquad \alpha_1 = 1,\ \alpha_2 = 1,\ \alpha_3 = -2$

$\hookrightarrow$ $[V] = L^1 T^{-1}$ $\qquad \alpha_1 = 1,\ \alpha_2 = -1$

$\hookrightarrow$ $[A] = L^1 T^{-2}$ $\qquad \alpha_1 = 1,\ \alpha_2 = -2$

- The invariance of the model under a change in units

$$d_j \longrightarrow \lambda_j d_j \qquad \text{for any } \lambda_j > 0$$

implies that:

$$a = d_1^{\alpha_1} d_2^{\alpha_2} \ldots d_r^{\alpha_r} \longrightarrow \lambda_1^{\alpha_1} d_1^{\alpha_1} \lambda_2^{\alpha_2} d_2^{\alpha_2} \ldots \lambda_r^{\alpha_r} d_2^{\alpha_2} = \lambda_1^{\alpha_1} \lambda_2^{\alpha_2} \ldots \lambda_r^{\alpha_r}\, a$$

$\hookrightarrow$ a direct consequence of this is that any two quantities that are equal must have the same dimension

- Thanks to scale-invariance, we can reduce the number of quantities appearing in a problem by introducing dimensionless quantities.

$\hookrightarrow$ In general, non-dimensional quantities can be interpreted as the ratio of two quantities with the same dimension appearing in the system, such as lengths, times, etc.

For example:

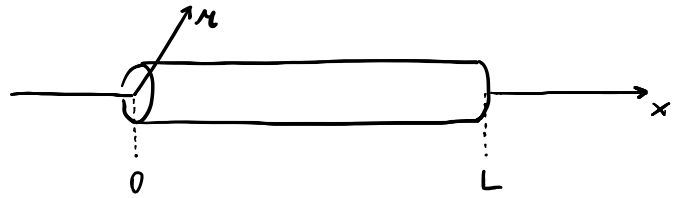
dimensional

$$\hat{x} = \frac{x}{L}$$

non-dimensional variable x

dimensional length of pipe

$[x] = [L] = L \qquad [\hat{x}] = 1$



$x \in [0, L]$ ← location along the length of the pipe

$\hat{x} \in [0, 1]$

- Any two systems with the same values of dimensionless parameters behave in the same way, up to rescaling.
- The introduction of dimensionless quantities reduces the number of variables in the problem (by the number of fundamental units).

$\Downarrow$

Let's see how the process of non-dimensionalizing works on a familiar example (hopefully!) and then we'll use it for one of the fundamental equations of fluid mechanics

ADVECTION - DIFFUSION EQUATION

- The advection-diffusion (transport) equation is one of the fundamental equations of fluid mechanics and describes the spreading of a dissolved species in a pipe.
  - dye, chemical pollutant, etc.

$$\frac{\partial c}{\partial t} = k\frac{\partial^2 c}{\partial x^2} - u\frac{\partial c}{\partial x} + f$$

$c(x=0,t) = C_0$ $\quad$ $c(x=L,t) = 0$

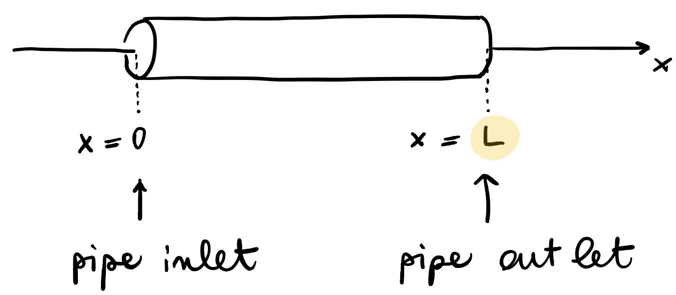
boundary conditions

$c(x,t)$: concentration of dissolved chemical

$\rightarrow [C] = \frac{mol}{vol} = \frac{mol}{m^3}$

x: spatial variable describing the location along the length of the pipe $\rightarrow [x] = m$

t : time $\rightarrow [t] = s$



parameters : k = diffusion coefficient of the chemical in the pipe

u = speed at which the water is flowing through the pipe

f = (constant) source of chemical that is leaking into the pipe from the walls

L = length of the pipe

C_0 = concentration of chemical coming into the pipe at the inlet

Note : can also indicate the partial derivatives as

$$\frac{\partial c}{\partial t} = c_t \qquad \frac{\partial^2 c}{\partial x^2} = c_{xx}$$

- Units for all parameters :

$[c] = \frac{mol}{m^3}$ $\qquad \frac{\partial c}{\partial t} = k \frac{\partial^2 c}{\partial x^2} - u \frac{\partial c}{\partial x} + f$

$[t] = s$ $\qquad c(x=0,t) = c_0 \qquad c(x=L,t) = 0$

$[x] = m$ $\qquad \left[\frac{\partial c}{\partial t}\right] = \frac{mol}{m^3} \frac{1}{s}$

$$\left[\frac{\partial c}{\partial x}\right] = \frac{mol}{m^3} \cdot \frac{1}{m} = \frac{mol}{m^4} \quad , \quad \left[\frac{\partial^2 c}{\partial x^2}\right] = \frac{mol}{m^3} \frac{1}{m^2} = \frac{mol}{m^5}$$

$$\left[\frac{\partial c}{\partial t}\right] = \left[k \frac{\partial^2 c}{\partial x^2}\right] - \left[u \frac{\partial c}{\partial x}\right] + [f]$$

$$\frac{\cancel{mol}}{\cancel{m^3} \cdot s} = [k] \frac{\cancel{mol}}{\cancel{m^5} \, 2} \longrightarrow [k] = \frac{m^2}{s}$$

$$\frac{\cancel{mol}}{\cancel{m^3} \cdot s} = [u] \frac{\cancel{mol}}{\cancel{m^4}} \longrightarrow [u] = \frac{m}{s}$$

$$\frac{mol}{m^3 \cdot s} = [f]$$

$$[c] = [c_0]$$
$$\downarrow$$
$$\frac{mol}{m^3} = [c_0]$$
$$m = [L]$$

- Non-dimensionalize the variables :

$\hat{c} = \frac{c}{c_0} \qquad \hat{x} = \frac{x}{L} \qquad \hat{t} = \frac{t}{t^*}$

given by the problem

leave t^* unchosen for now

$[\hat{c}] = [\hat{x}] = [\hat{t}] = 1$

- Plug this in the problem :

$$\frac{\partial c}{\partial t} = k \frac{\partial^2 c}{\partial x^2} - u \frac{\partial c}{\partial x} + f$$

$$c(x=0,t) = c_0 \qquad c(x=L,t) = 0$$

$$\frac{\partial(\hat{c}C_0)}{\partial(\hat{t}t^*)} = k\,\frac{\partial^2(\hat{c}C_0)}{\partial(\hat{x}L)^2} - u\,\frac{\partial(\hat{c}C_0)}{\partial(\hat{x}L)} + f$$

$$\hat{c}\cancel{C_0}(\hat{x}L = 0, \hat{t}t^*) = \cancel{C_0} \longrightarrow \hat{c}(\hat{x} = 0, \hat{t}t^*) = 1$$

$$\hat{c}C_0(\hat{x}\cancel{L} = \cancel{L}, \hat{t}t^*) = 0 \longrightarrow \hat{c}(\hat{x} = 1, \hat{t}t^*) = 0$$

$$\frac{C_0}{t^*}\frac{\partial\hat{c}}{\partial\hat{t}} = k\,\frac{C_0}{L^2}\frac{\partial^2\hat{c}}{\partial\hat{x}^2} - u\,\frac{C_0}{L}\frac{\partial\hat{c}}{\partial\hat{x}} + f$$

$$\frac{\partial\hat{c}}{\partial\hat{t}} = k\,\frac{t^*}{\cancel{C_0}}\frac{\cancel{C_0}}{L^2}\frac{\partial^2\hat{c}}{\partial\hat{x}^2} - u\,\frac{t^*}{\cancel{C_0}}\frac{\cancel{C_0}}{L}\frac{\partial\hat{c}}{\partial\hat{x}} + \frac{t^*}{C_0}f$$

$$\frac{\partial\hat{c}}{\partial\hat{t}} = \underbrace{\frac{k\,t^*}{L^2}}\frac{\partial^2\hat{c}}{\partial\hat{x}^2} - \underbrace{\frac{u\,t^*}{L}}\frac{\partial\hat{c}}{\partial\hat{x}} + \frac{t^*}{C_0}f$$

candidates for t^* — what shall we choose?

let's define: $t_{diff} = \frac{L^2}{k} \longrightarrow \frac{k}{L^2}t_{diff} = 1$

$t_{adv} = \frac{L}{u} \longrightarrow \frac{u}{L}t_{adv} = 1$

then we have

$$\text{Péclet number} = \frac{t_{diff}}{t_{adv}} = \frac{L^{\cancel{2}}}{k}\cdot\frac{u}{\cancel{L}} = \frac{Lu}{k} = Pe$$

this non-dimensional parameter shows which of the two effects between advection and diffusion is most important in the system of interest

↳ specifically, we're comparing time-scales, i.e. whether diffusion happens more slowly than advection or vice-versa (and by how much)