# Probabilistic implications of a simple formula: $\sin(2x) = 2\sin(x)\cos(x)$

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# Vietta formula

$$\begin{aligned}\sin(x) &= 2\sin(x/2)\cos(x/2)\\ &= 2^2\sin(x/2^2)\cos(x/2^2)\cos(x/2)\\ &= \cdots\\ &= 2^N\sin(x/2^N)\prod_{n=1}^{N}\cos(x/2^n).\end{aligned}$$

# Vietta formula

$$\frac{\sin(x)}{x} = \frac{\sin(x/2^N)}{x/2^N} \prod_{n=1}^{N} \cos(x/2^n).$$

# Vietta formula

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$$

$$\frac{\sin(x)}{x} = \prod_{i=1}^{\infty} \cos(x/2^i).$$

# Cool observation

$$\frac{2}{\pi} = ?$$

# Another look at the Vietta formula

## Binary expansion of real numbers

$$t = \sum_{n=1}^{\infty} \frac{a_n(t)}{2^n} \in [0, 1]$$

$a_n = 0$ or $1$
Is $a_n(t)$ a function?
Uniqueness issue: $\frac{3}{4} : (1, 0, 1, 1, ...)$ or $(1, 1, 0, 0, ...)$

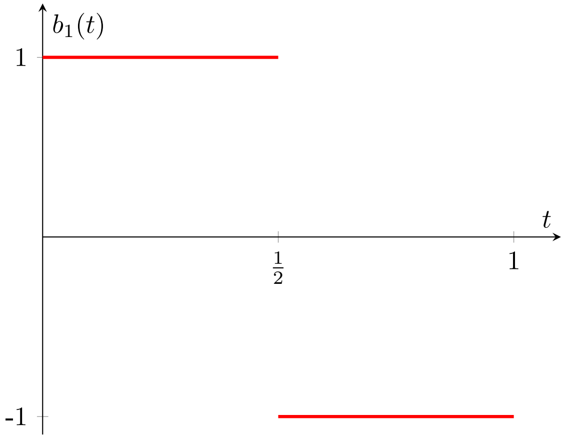
# Another look at the Vietta formula

## Convenient change of variables

$$1 - 2t = \sum_{n=1}^{\infty} \frac{b_n(t)}{2^n} \in [0, 1]$$

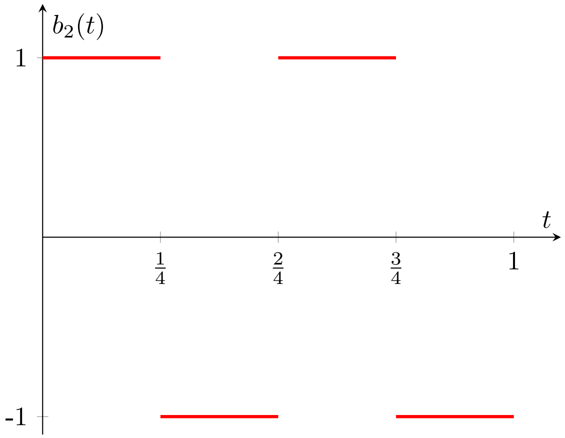
$b_n = -1$ or $1$

# Another look at the Vietta formula



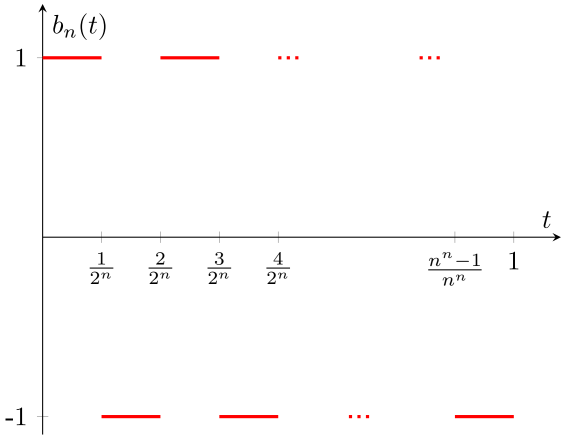
# Another look at the Vietta formula

$b_n(t)$



# Another look at the Vietta formula

$b_n(t)$



# Another look at the Vietta formula

## De Moivre

$$e^{i\theta} = \cos(\theta) + i\sin(\theta), \quad i = \sqrt{-1}.$$

$$\frac{\sin x}{x} = \int_0^1 e^{ix(1-2t)} dt \qquad \cos(x/2^n) = \int_0^1 e^{ixb_n/2^n} dt.$$

$$\int_0^1 e^{ix\sum_{n=1}^{\infty} \frac{b_n(t)}{2^n}} dt = \prod_{n=1}^{\infty} \int_0^1 e^{ixb_n(t)/2^n} dt$$

# Another look at the Vietta formula

Switch of order between $\prod$ and $\int$

$$\int_0^1 \prod_{n=1}^{\infty} e^{ix\frac{b_n(t)}{2^n}}\,dt = \prod_{n=1}^{\infty} \int_0^1 e^{ixb_n(t)/2^n}\,dt$$

Coincidence or something of substance?

# Coin toss probability

## Infinite trials

$$\{(\omega_1, \omega_2, ...)\ :\ \omega_n = H \quad \text{or} \quad T\}$$

For example, $(H, H, T, T, ...)$.

# Coin toss probability

## Random numbers in $[0, 1)$

$$(\omega_1, \omega_2, \ldots) \mapsto t = \sum_{n=1}^{\infty} \frac{a_n(t)}{2^n} \in [0, 1]$$

$a_n = 0$ if $\omega_n = T$ and $a_n = 1$ if $\omega_n = H$.
Uniqueness issue: Is the function one to one?

# Coin toss probability

## Infinite trials

$$\{(\omega_1, \omega_2, \ldots) \ : \ \omega_n = H \quad \text{or} \quad T\}$$

For example, $(H, H, T, T, \ldots)$.

# Coin toss probability

$$b_n(t) = 1 - 2a_n(t)$$

# Thank you!

Q& A!