Solitons: Why isn't that wave breaking?

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Outline

- John Scott Russell and the “Wave of Translation”
- Travelling Waves
- Nonlinear Steepening and Wave Breaking
- Dispersion and wave spreading
- Zabusky, Kruskal, and the KdV equation

John Scott Russell

“Wave of Translation”

- In 1834, while experimenting on efficient canal boat designs in the Union Canal, John Scott Russell noticed something peculiar:

  I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed.

  - John Scott Russell

- He followed the wave for one to two miles before eventually losing it.
- He termed this wave the “Wave of Translation”. We know it today as a soliton.

“Wave of Translation”

"Wave of Translation"

"Wave of Translation"

- So what exactly was going on? How did the wave keep its shape so long? Why didn't the wave break? Why did it take so long to flatten out?
- Before we delve into this, we need to define some terms and do some math.

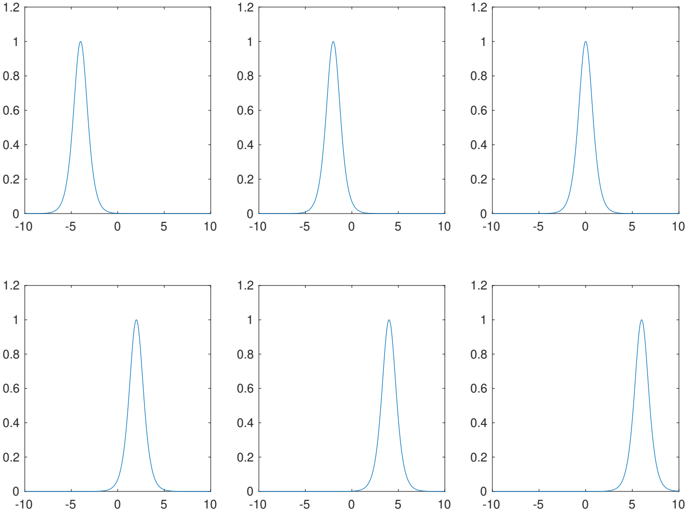
Travelling Wave

- Consider the linear transport equation

$$u_t - cu_x = 0, \qquad c = \text{constant}$$

- The general solution is given by $u(x,t) = f(x - ct)$ for f an arbitrary function (determined by some initial condition).
- This describes a wave travelling to the right with constant speed c.
- Example: Take $u(x,0) = f(x) = \text{sech}^2(x)$. Then the solution looks like: $u(x,t) = \text{sech}^2(x - ct)$.
- The ct part essentially acts like a horizontal translation to the right.

Travelling Waves



Nonlinear Steepening and Wave Breaking

- What makes a wave break?
  - Answer: Nonlinearity!
- To see this, we'll consider the following nonlinear transport model:

$$u_t + uu_x = 0, \qquad u(x,0) = f(x)$$

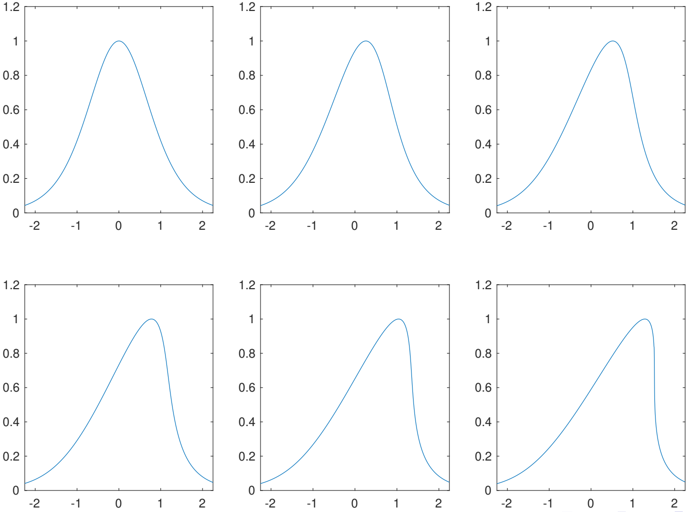
- This equation is more commonly known as Inviscid Burgers' Equation.
- The method of characteristics can be used to solve this equation, leading to the implicit solution

$$u(x,t) = f(x - ut)$$

- This describes a travelling wave moving to the right at with speed u (not constant!). That is to say, points with larger u-value will travel faster.

Nonlinear Steepening and Wave Breaking

- Example: $u_t + uu_x = 0$, $u(x,0) = \text{sech}^2(x)$.



Nonlinear Steepening and Wave Breaking

- No sugarcoating it, this is a problem. The wave steepens to a point where it becomes nearly vertical (the magnitude of the derivative $|u_x|$ tends to ∞) and the peak overtakes the trough.
    - As a result the wave 'breaks'.
- Mathematically speaking, we'll end up with a multi-valued function unless one fixes this by allowing the introduction of a discontinuity into the solution (leading to the formation of a shock).
- But this is not the only solution. There are other ways to 'fix' the issue of wave breaking.
    - Dispersion to the rescue!
    - So what exactly is dispersion?

Dispersion

- Basic Idea: Waves with different wavelengths travel at different speeds.
- As a concrete example, we'll consider the following equation:

$$u_t + u_{xxx} = 0 \qquad \text{(linear KdV)}$$

- Look for a travelling wave solution of the form: $u = A\cos(kx - \omega t)$. Here k is the wave number (inversely proportional to wavelength) and ω is the frequency.
- Plugging this in leads to the (dispersion) relation: $\omega = k^3$, so that $u(x,t) = A\cos(kx - k^3 t)$.
- This is also a travelling wave with speed $\frac{\omega}{k} = k^2$. That is, the speed of the wave depends on the wave number.
- Generally, an initial condition $u(x,0)$ will be represented as a linear combination of different modes (cosines with different k).
- These will travel at different speeds, leading to the solution spreading or smoothing out.

Solitons: A perfect balance

- Solitons represent a balance between nonlinearity and dispersion.

- Put the nonlinearity and dispersion together:

$$u_t + uu_x + \delta^2 u_{xxx} = 0$$

- Zabusky and Kruskal (1965) found that the combination led to a competition between the dispersion and the nonlinearity.
- The wave initially steepens, then begins to spread, leading to the formation of a soliton 'train' composed of permanent form waves which propagate without loss of shape.

Different Flavors of Solitons

- Solitons have been found in many forms and for many systems.
- Systems include: KdV, NLS, KP Equation, DS Equations, Camassa-Holm Equation, etc.
- Different types of solitons include: (regular) soliton, cuspon, peakon, compacton, loop soliton, vortex soliton (look up f alaco soliton), dromion (2D), etc.

Thank You