

Riemannian Optimization and a Riemannian Proximal Newton-CG Method

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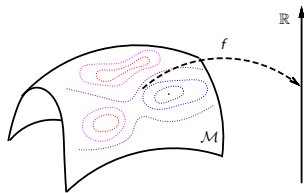
- Riemannian optimization;
 - Problem statement;
 - Applications;
 - Smooth optimization framework;
 - Research foci of Riemannian optimization;
- A Riemannian proximal Newton-CG method;
 - Optimization with a structure;
 - Proximal gradient-type methods;
 - A Riemannian proximal Newton method;
 - Globalization by truncated CG;
 - Numerical experiments;
- Summary;

Riemannian Optimization

Problem: Given $f(x) : \mathcal{M} \rightarrow \mathbb{R}$,
solve

$$\min_{x \in \mathcal{M}} f(x)$$

where \mathcal{M} is a Riemannian manifold.

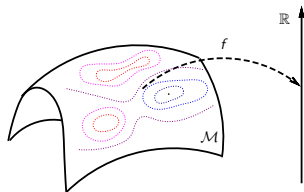


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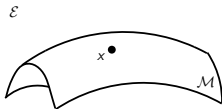
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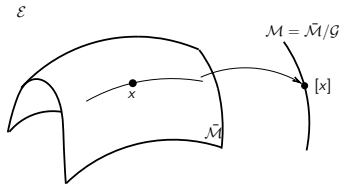


Two kinds of commonly-encountered manifolds

Embedded submanifold of a Euclidean space



Quotient manifold from an embedded submanifold

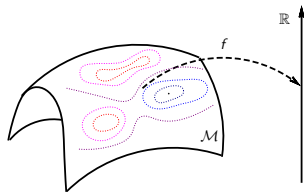


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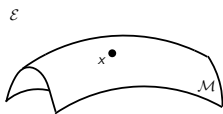
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Examples:

- Sphere: $\{x \in \mathbb{R}^n \mid \|x\| = 1\}$;
- Stiefel manifold:
 $\text{St}(p, n) = \{X \in \mathbb{R}^{n \times p} \mid X^T X = I_p\}$;
- Fixed rank:
 $\mathbb{R}_r^{m \times n} = \{X \in \mathbb{R}^{m \times n} : \text{rank}(X) = r\}$;
- etc;

Embedded submanifold of a Euclidean space

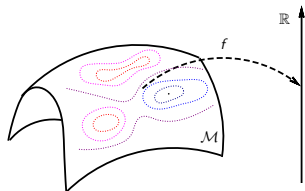


Riemannian Optimization

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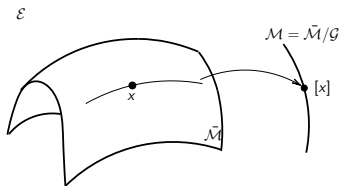
where \mathcal{M} is a Riemannian manifold.



Examples:

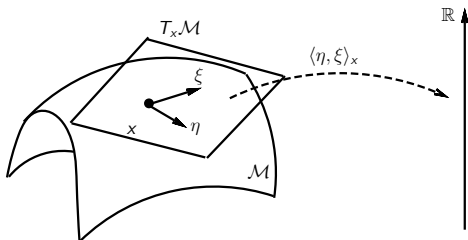
- Grassmann manifold:
the set of p dimensional linear
spaces in \mathbb{R}^n
 $\text{Gr}(p, n) = \text{St}(p, n) / \mathcal{O}_p$
- Shape space;
- etc;

Quotient manifold from an embedded submanifold



Riemannian Optimization

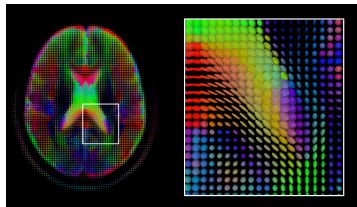
Roughly, a Riemannian manifold \mathcal{M} is a smooth set with a smoothly-varying inner product on the tangent spaces.



Riemannian manifold = Manifold + Riemannian metric (inner products)

Embedded submanifold: Computation on SPD manifold

- SPD manifold:
 $\mathcal{S}_{++}^n = \{X \in \mathbb{R}^{n \times n} : X = X^T, X \succ 0\};$
- Applications of SPD matrices
 - Diffusion tensors in medical imaging [CSV12, FJ07, RTM07]
 - Describing images and video [LWM13, SFD02, ASF⁺05, TPM06, HWSC15]
- Motivation of averaging SPD matrices
 - denoising / interpolation
 - clustering / classification



Embedded submanifold: Computation on SPD manifold

One averaging SPD matrices method:

$$G(A_1, \dots, A_k) = \arg \min_{X \in \mathcal{S}_{++}^n} \frac{1}{2k} \sum_{i=1}^k \text{dist}^2(X, A_i),$$

where $\text{dist}(X, Y) = \|\log(X^{-1/2} Y X^{-1/2})\|_F$ is the distance under the Riemannian metric $\langle \eta_X, \xi_X \rangle_X = \text{trace}(\eta_X X^{-1} \xi_X X^{-1})$.

Embedded submanifold: Computation on SPD manifold

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Why shall we use Riemannian optimization approach?

Metric: $\langle \eta_X, \xi_X \rangle_X = \text{trace}(\eta_X X^{-1} \xi_X X^{-1})$

Metric: $\langle \eta, \xi \rangle_X = \text{trace}(\eta^T \xi)$

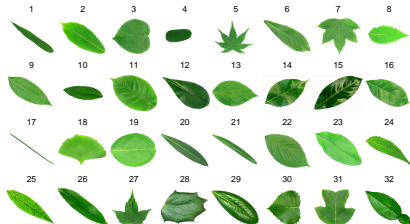
Condition number of the Riemannian Hessian [YHAG2020]

- $\kappa(H^R) \leq 1 + \frac{\ln(\max \kappa_i)}{2}$, where
 $\kappa_i = \kappa(\mu^{-1/2} A_i \mu^{-1/2})$
- $\kappa(H^R) \leq 20$ if $\max(\kappa_i) = 10^{16}$

- $\frac{\kappa^2(\mu)}{\kappa(H^R)} \leq \kappa(H^E) \leq \kappa(H^R) \kappa^2(\mu)$
- $\kappa(H^E) \geq \kappa^2(\mu)/20$

[YHAG2020]: X. Yuan, W. Huang*, P.-A. Absil, K. A. Gallivan. "Computing the matrix geometric mean: Riemannian vs Euclidean conditioning, implementation techniques, and a Riemannian BFGS method", *Numerical Linear Algebra with Applications*, 27:5, 1-23, 2020.

Quotient manifold: Computation on shape space



- Classification [LKS⁺12, HGSA15]
- Face recognition [DBS⁺13]



Quotient manifold: Computation on shape space

- Elastic shape analysis invariants:
 - Rescaling
 - Translation
 - Rotation
 - Reparametrization
- The shape space is a quotient space

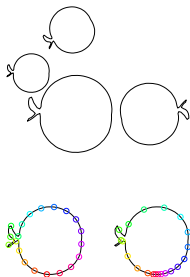
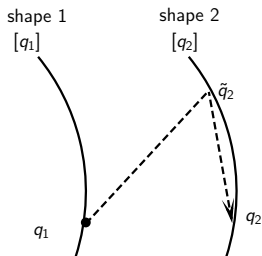


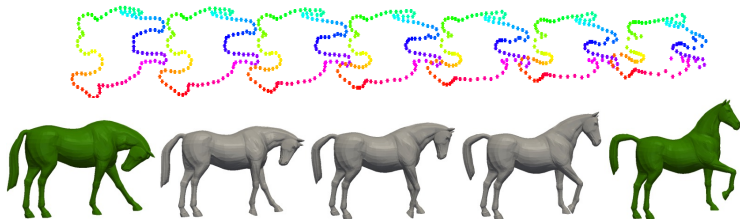
Figure: All are the same shape.

Quotient manifold: Computation on shape space Registration



- Optimization problem $\min_{q_2 \in [q_2]} \text{dist}(q_1, q_2)$ is defined on a Riemannian manifold

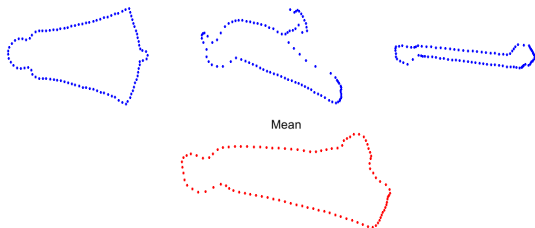
Quotient manifold: Computation on shape space Geodesic / Interpolation



$$\min_{\alpha \in \mathcal{H}_{x,y}} \frac{1}{2} \int_0^1 \langle \dot{\alpha}(\tau), \dot{\alpha}(\tau) \rangle_{\alpha(\tau)} d\tau$$

- Computation of a geodesic between two shapes
- Interpolation in shape space

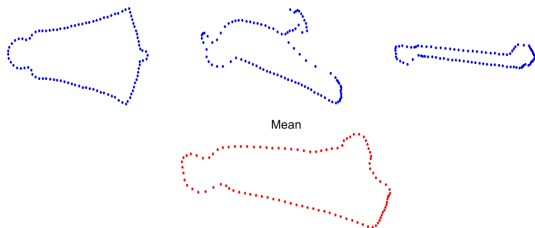
Quotient manifold: Computation on shape space Karcher mean



$$\min_{x \text{ is a shape}} \frac{1}{2k} \sum_{i=1}^k \text{dist}^2(X, S_i),$$

- Computation of Karcher mean of a population of shapes

Quotient manifold: Computation on shape space Karcher mean



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Riemannian optimization is used since these problems naturally involve a Riemannian manifold

Smooth Optimization Framework

Iterations on the Manifold

Consider the following generic update for an iterative Euclidean optimization algorithm:

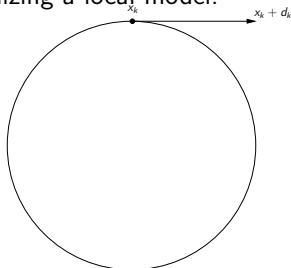
$$x_{k+1} = x_k + \Delta x_k = x_k + \alpha_k s_k .$$

This iteration is implemented in numerous ways, e.g.:

- Steepest descent: $x_{k+1} = x_k - \alpha_k \nabla f(x_k)$
- Newton's method: $x_{k+1} = x_k - [\nabla^2 f(x_k)]^{-1} \nabla f(x_k)$
- Trust region method: Δx_k is set by optimizing a local model.

Riemannian Manifolds Provide

- Riemannian concepts describing **directions** and **movement** on the manifold
- Riemannian analogues for **gradient** and **Hessian**



Smooth Optimization Framework

Riemannian gradient and Riemannian Hessian

Definition

The **Riemannian gradient** of f at x is the unique tangent vector in $T_x \mathcal{M}$ satisfying $\forall \eta \in T_x \mathcal{M}$, the directional derivative

$$Df(x)[\eta] = \langle \text{grad } f(x), \eta \rangle$$

and $\text{grad } f(x)$ is the direction of steepest ascent.

Definition

The **Riemannian Hessian** of f at x is a symmetric linear operator from $T_x \mathcal{M}$ to $T_x \mathcal{M}$ defined as

$$\text{Hess } f(x) : T_x \mathcal{M} \rightarrow T_x \mathcal{M} : \eta \rightarrow \nabla_{\eta} \text{grad } f,$$

where ∇ is the affine connection.

Smooth Optimization Framework

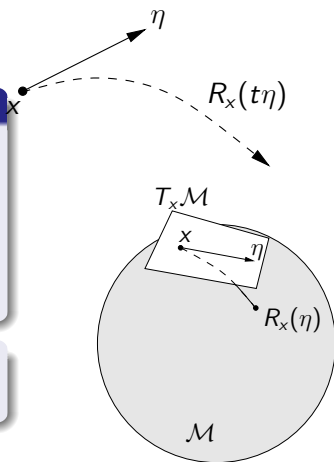
Retractions

Euclidean	Riemannian
$x_{k+1} = x_k + \alpha_k d_k$	$x_{k+1} = R_{x_k}(\alpha_k \eta_k)$

Definition

A **retraction** is a mapping R from $T\mathcal{M}$ to \mathcal{M} satisfying the following:

- R is continuously differentiable
 - $R_x(0) = x$
 - $DR_x(0)[\eta] = \eta$
-
- maps tangent vectors back to the manifold
 - defines curves in a direction



Smooth Optimization Framework

Categories of Riemannian smooth optimization methods

Retraction-based: local information only

Line search-based: use local tangent vector and $R_x(t\eta)$ to define line

- Steepest decent
- Newton

Local model-based: series of flat space problems

- Riemannian trust region Newton (RTR)
- Riemannian adaptive cubic overestimation (RACO)

Smooth Optimization Framework

Categories of Riemannian smooth optimization methods

Retraction and transport-based: information from multiple tangent spaces

- Nonlinear conjugate gradient: multiple tangent vectors
- Quasi-Newton e.g. Riemannian BFGS: transport operators between tangent spaces

Additional element required for optimizing a cost function;

- formulas for combining information from multiple tangent spaces.

Smooth Optimization Framework

Categories of Riemannian smooth optimization methods

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Vector Transport:

- Vector transport: Transport a tangent vector from one tangent space to another;
- $\mathcal{T}_{\eta_x} \xi_x$, denotes transport of ξ_x to tangent space of $R_x(\eta_x)$. R is a retraction associated with \mathcal{T} ;

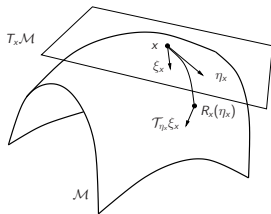


Figure: Vector transport.

Smooth Optimization Framework

Retraction/Transport-based Riemannian optimization

Given a retraction and a vector transport, we can generalize classical unconstrained smooth optimization methods from Euclidean space to the Riemannian manifold.

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Given a retraction and a vector transport, we can generalize classical unconstrained smooth optimization methods from Euclidean space to the Riemannian manifold.

Do the Riemannian versions of those methods work well?

Smooth Optimization Framework

Retraction/Transport-based Riemannian optimization

Given a retraction and a vector transport, we can generalize classical unconstrained smooth optimization methods from Euclidean space to the Riemannian manifold.

Do the Riemannian versions of those methods work well?

No, generally

- Lose many theoretical results and important properties;
- Impose restrictions on retraction/vector transport;

Research Foci of Riemannian Optimization

- 1 Manifold recognition, geometry structure analyses and computations;
 - 2 Generalization Euclidean algorithms to the Riemannian setting;
 - 3 Algorithms specialization for applications;
 - 4 Library developments;
-

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- Manifold recognition
 - Riemannian metric
 - Retraction / Geodesic
 - Vector transport / Parallel translation

[EAS1998] A. Edelman, T. A. Arias, and S. T. Smith. The geometry of algorithms with orthogonality constraints. *SIAM Journal on Matrix Analysis and Applications*, 20(2):303–353, 1998

[CMV2017] T Carson, D. G. Mixon, and S. Villar. Manifold optimization for k-means clustering. In *2017 International Conference on Sampling Theory and Applications (SampTA)*, 73–77. IEEE, 2017

[SDN2021] G. Song, W. Ding, and M. K. Ng, Low rank pure quaternion approximation for pure quaternion matrices, *SIAM Journal on Matrix Analysis and Applications*, 42, pp. 58–82, 2021

[VAV2013] B. Vandereycken, P.-A. Absil, and S. Vandewalle. A Riemannian geometry with complete geodesics for the set of positive semidefinite matrices of fixed rank, *IMA Journal of Numerical Analysis*, 33.2, 481–514, 2013.

[Zim2017] R. Zimmermann. A matrix-algebraic algorithm for the Riemannian logarithm on the Stiefel manifold under the canonical metric. *SIAM Journal on Matrix Analysis and Applications*, 38.2, 322–342, 2017.

Research Foci of Riemannian Optimization

- 1 Manifold recognition, geometry structure analyses and computations;
 - 2 Generalization Euclidean algorithms to the Riemannian setting;
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- Smooth unconstrained optimization algorithms
- Nonsmooth unconstrained optimization algorithms
- Constrained optimization algorithms

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Riemannian optimization mainly focuses on this topic.
Discuss later.

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- Computations on the SPD manifold;
- Computations on the shape space;
- Clustering and graph partitions;
- Beamforming in wireless communication;
- Blind source separation;
- etc

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- Representation of a manifold and tangent spaces;
- Choose a Riemannian metric;
- Choose a retraction;
- Choose a vector transport;

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Above factors may influence algorithms significantly.

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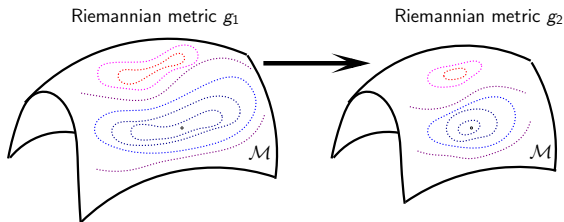


Figure: Changing Riemannian metric may influence the difficulty of a problem.

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- Manopt (Matlab library) [Boumal, Mishra, Absil, Sepulchre(2014)]
- Pymanopt (Python version of Manopt) [Townsend, Koep, Weichwald (2016)]
- Manoptjl (Julia, nonsmooth methods) [Bergmann (2019)]
- ROPTLIB (C++ library, interfaces to Matlab and Julia)
[Huang, Absil, Gallivan, Hand (2018)]
- ManifoldOptim (R wrapper of ROPTLIB) [Martin, Raim, Huang, Adraghi (2018)]
- McTorch (Python, GPU acceleration)
[Meghawanshi, Jawanpuria, Kunchukuttan, Kasai, Mishra (2018)]
- CDOpt (Python, embedded submanifold in the form of $c(x) = 0$)
[Xiao, Hu, Liu, Toh (2022)]

- 1 Manifold recognition, geometry structure analyses and computations;
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Provide theories to explain behaviors of existing algorithms for particular applications

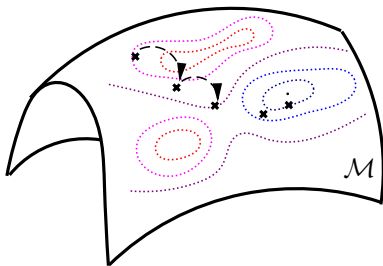
- [MBDG2023]: IRKA is a Riemannian gradient descent method;
- [YHAG2020]: Richardson-like iteration for matrix geometric mean is a Riemannian gradient descent method;
- [BM2006]: The improved BFGS method is a Riemannian BFGS method using vector transport by parallelization;

[MBDG2023] P. Mlinaric, C. Beattie, Z. Drmac, and S. Gugercin. IRKA is a Riemannian Gradient Descent Method. arxiv:2311.02031, 2023
[YHAG2020] X. Yuan, W. Huang, P.-A. Absil, K. A. Gallivan. Computing the matrix geometric mean: Riemannian vs Euclidean conditioning, implementation techniques, and a Riemannian BFGS method, *Numerical Linear Algebra with Applications*, 27:5, 1-23, 2020
[BM2006] I. Brace and J. H. Manton. An improved BFGS-on-manifold algorithm for computing weighted low rank approximations. *Proceedings of 17th international Symposium on Mathematical Theory of Networks and Systems*, P.1735–1738, 2006

Comparison with Constrained Optimization

Not all Riemannian optimization problem can be formulated as constrained optimization problems, and vice versa.

- All iterates on the manifold
- Convergence properties of unconstrained optimization algorithms
- No need to consider Lagrange multipliers or penalty functions
- Exploit the structure of the constrained set



A Non-exhaustive Review

- Smooth unconstrained problems
 - Steepest descent: Smith 1994; Helmke-Moore 1994; Iannazzo-Porcelli 2019;
 - Conjugate gradient: Smith 1994; Gallivan-Absil 2010; Ring-Wirth 2012; Sato-Iwai 2015;
 - Quasi-Newton: Ring-Wirth 2012; Huang-Absil-Gallivan 2018; Huang-Gallivan 2022
 - Newton-CG: Absil-Baker-Gallivan 2007; Huang-Huang 2023
- Nonsmooth unconstrained problems
 - Proximal point method: Ferreira-Oliveira 2002;
 - Optimality conditions: Yang-Zhang-Song 2014;
 - Gradient sampling: Huang 2013; Hosseini and Uschmajew 2017;
 - ϵ -subgradient-based methods: Grohs-Hosseini 2015;
 - Proximal gradient methods: Huang-Wei 2022;
 - Proximal Newton method: Si-Absil-Huang-Jiang-Vary 2023;
- Constrained problems:
 - Augmented Lagrangian methods: Boumal-Liu 2019;
 - Sequential quadratic programming: Obara-Okuno-Takeda 2022;
 - Frank-Wolfe Methods: Weber-Sra 2023;

A Non-exhaustive Review

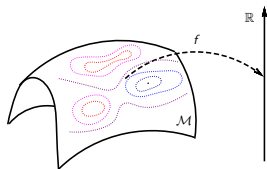
- Smooth unconstrained problems:
 - Stiefel manifold: Wen-Yin 2012; Jiang-Dai 2014; Xiao-Liu-Yuan 2020; Dai-Wang-Zhou 2020
 - Symplectic Stiefel manifold: Gao-Son-Absil-Stykel 2021
 - Symmetric positive definite manifold: Bini-Iannazzo 2013; Zhang 2017; Yuan-Huang-Absil-Gallivan 2020;
 - Fixed rank manifold: Wen-Yin-Zhang 2012; Mishra 2014; Sutti-Vandereycken 2021; Levin-Kileel-Boumal 2022
- Nonsmooth unconstrained problems:
 - Stiefel Manifold: Huang-Wei 2019; Chen-Ma-So-Zhang 2020; Xiao-Liu-Yuan 2020;
 - Fixed rank manifold: Cambier-Absil 2016;
 - Matrix manifolds: Zhou-Bao-Ding-Zhu 2022
 - Smooth equation constraints: Xiao-Liu-Toh 2023
- Constrained problems:
 - Stiefel + non-negativity: Jiang-Meng-Wen-Chen 2019;
 - Symmetric positive definite + zeros: Phan-Menickelly 2020;
 - Low-rank manifold + structured Linear: Yang-Gao-Yuan 2025;

A Riemannian Proximal Newton-CG Method

Problem statement

Optimization on Manifolds with Structure:

$$\min_{x \in \mathcal{M}} F(x) = f(x) + h(x),$$



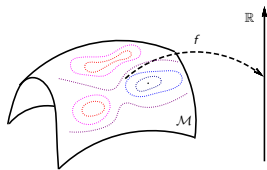
- \mathcal{M} is a finite-dimensional Riemannian manifold;
- f is smooth and may be nonconvex; and
- $h(x)$ is continuous and convex but may be nonsmooth;

A Riemannian Proximal Newton-CG Method

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Optimization on Manifolds with Structure:

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- f is smooth and may be nonconvex; and
- $h(x)$ is continuous and convex but may be nonsmooth;

Applications: sparse PCA [ZHT06], compressed modes [OLCO13], sparse partial least squares regression [CSG⁺18], sparse inverse covariance estimation [BESS19], sparse blind deconvolution [ZLK⁺17], and clustering [HWGVD22].

A Riemannian Proximal Newton-CG Method

Euclidean proximal gradient/Newton method

Optimization with Structure: $\mathcal{M} = \mathbb{R}^n$

$$\min_{x \in \mathbb{R}^n} F(x) = f(x) + h(x),$$

Given x_0 ,

$$\begin{cases} d_k = \arg \min_{p \in \mathbb{R}^n} f(x_k) + \langle \nabla f(x_k), p \rangle + \frac{1}{2} \langle p, H_k p \rangle + h(x_k + p) \\ x_{k+1} = x_k + d_k. \end{cases}$$

A Riemannian Proximal Newton-CG Method

Euclidean proximal gradient/Newton method

Optimization with Structure: $\mathcal{M} = \mathbb{R}^n$

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proximal gradient: $H_k = L I_n$

- $h \equiv 0 \Rightarrow$ Steepest descent;
- Linear convergence;

proximal Newton: $H_k = \nabla^2 f(x_k)$

- $h \equiv 0 \Rightarrow$ Newton;
- Superlinear convergence;

A Riemannian Proximal Newton-CG Method

Euclidean proximal gradient/Newton method

Optimization with Structure: $\mathcal{M} = \mathbb{R}^n$

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Given x_0 ,

$$\begin{cases} d_k = \arg \min_{p \in \mathbb{R}^n} f(x_k) + \langle \nabla f(x_k), p \rangle + \frac{1}{2} \langle p, H_k p \rangle + h(x_k + p) \\ x_{k+1} = x_k + d_k. \end{cases}$$

proximal gradient: $H_k = L I_n$

- $h \equiv 0 \Rightarrow$ Steepest descent;
- Linear convergence;

proximal Newton: $H_k = \nabla^2 f(x_k)$

- $h \equiv 0 \Rightarrow$ Newton;
- Superlinear convergence;

How to generalize to the Riemannian setting?

A Riemannian Proximal Newton-CG Method

Generalizations of proximal gradient method

Euclidean Proximal gradient:

Given x_0 ,

$$\begin{cases} d_k = \arg \min_{p \in \mathbb{R}^n} f(x_k) + \langle \nabla f(x_k), p \rangle + \frac{1}{2} \langle p, p \rangle + h(x_k + p) \\ x_{k+1} = x_k + d_k. \end{cases}$$

Riemannian generalization 1: (for embedded submanifold)

$$\left. \begin{array}{l} \nabla f(x_k) \implies \text{grad } f(x_k) \\ x_{k+1} = x_k + d_k \implies x_{k+1} = R_{x_k}(d_k) \\ p \in \mathbb{R}^n \implies p \in T_{x_k} \mathcal{M} \end{array} \right\} \implies \text{Converge globally}$$

$$\begin{cases} d_k = \arg \min_{p \in T_{x_k} \mathcal{M}} f(x_k) + \langle \text{grad } f(x_k), p \rangle + \frac{1}{2} \langle p, p \rangle + h(x_k + p) \\ x_{k+1} = R_{x_k}(d_k). \end{cases}$$

A Riemannian Proximal Newton-CG Method

Generalizations of proximal gradient method

Euclidean Proximal gradient:

Given x_0 ,

$$\begin{cases} d_k = \arg \min_{p \in \mathbb{R}^n} f(x_k) + \langle \nabla f(x_k), p \rangle + \frac{L}{2} \langle p, p \rangle + h(x_k + p) \\ x_{k+1} = x_k + d_k. \end{cases}$$

Riemannian generalization 2: (for general manifold)

$$\left. \begin{array}{l} \nabla f(x_k) \implies \text{grad } f(x_k) \\ x_{k+1} = x_k + d_k \implies x_{k+1} = R_{x_k}(d_k) \\ p \in \mathbb{R}^n \implies p \in T_{x_k} \mathcal{M} \\ h(x_k + p) \implies h(R_{x_k}(p)) \end{array} \right\} \implies \begin{array}{l} \text{Converge globally} \\ \text{Convergence rate analyses} \end{array}$$

$$\begin{cases} d_k = \arg \min_{p \in T_{x_k} \mathcal{M}} f(x_k) + \langle \text{grad } f(x_k), p \rangle + \frac{L}{2} \langle p, p \rangle + h(R_{x_k}(p)) \\ x_{k+1} = R_{x_k}(d_k). \end{cases}$$

A Riemannian Proximal Newton-CG Method

A native generalization

Euclidean proximal Newton:

$$\begin{cases} d_k = \operatorname{argmin}_{p \in \mathbb{R}^n} f(x_k) + \langle \nabla f(x_k), p \rangle + \frac{1}{2} \langle p, \nabla^2 f(x_k) p \rangle + h(x_k + p) \\ x_{k+1} = x_k + d_k \end{cases}$$

A native generalization by replacing the Euclidean gradient and Hessian by the Riemannian gradient and Hessian:

$$\begin{cases} \eta_k = \operatorname{arg min}_{\eta \in \mathbb{T}_{x_k} \mathcal{M}} f(x_k) + \langle \operatorname{grad} f(x_k), \eta \rangle + \frac{1}{2} \langle \eta, \operatorname{Hess} f(x_k) \eta \rangle + h(x_k + \eta) \\ x_{k+1} = R_{x_k}(\eta_k) \end{cases}$$

A Riemannian Proximal Newton-CG Method

A native generalization

Euclidean proximal Newton:

$$\begin{cases} d_k = \operatorname{argmin}_{p \in \mathbb{R}^n} f(x_k) + \langle \nabla f(x_k), p \rangle + \frac{1}{2} \langle p, \nabla^2 f(x_k) p \rangle + h(x_k + p) \\ x_{k+1} = x_k + d_k \end{cases}$$

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Does it converge superlinearly locally?

A Riemannian Proximal Newton-CG Method

A native generalization

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$$\begin{cases} d_k = \operatorname{argmin}_{p \in \mathbb{R}^n} f(x_k) + \langle \nabla f(x_k), p \rangle + \frac{1}{2} \langle p, \nabla^2 f(x_k) p \rangle + h(x_k + p) \\ x_{k+1} = x_k + d_k \end{cases}$$

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Does it converge superlinearly locally?

Not necessarily!

A Riemannian Proximal Newton-CG Method

A native generalization

Consider the Sparse PCA over sphere:

$$\min_{x \in \mathbb{S}^{n-1}} -x^T A^T A x + \mu \|x\|_1,$$

where $f(x) = -x^T A^T A x$, $h(x) = \mu \|x\|_1$.

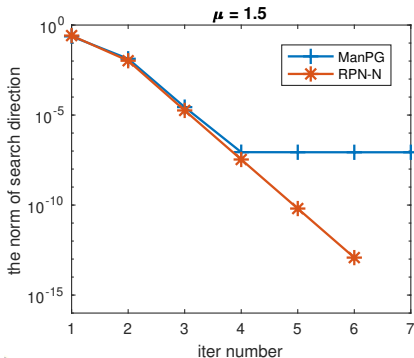


Figure: Comparisons of native generalization (RPN-N) and the proximal gradient method (ManPG) in [CMSZ20].

A Riemannian Proximal Newton-CG Method

A native generalization

Euclidean version:

$$\begin{cases} d_k = \operatorname{argmin}_p \langle \nabla f(x_k), p \rangle + \frac{1}{2} \langle p, \nabla^2 f(x_k) p \rangle + h(x_k + p) \\ x_{k+1} = x_k + d_k \end{cases}$$

A native generalization by replacing the Euclidean gradient and Hessian by the Riemannian gradient and Hessian:

$$\begin{cases} \eta_k = \operatorname{arg min}_{\eta \in T_{x_k}} \mathcal{M} f(x_k) + \langle \operatorname{grad} f(x_k), \eta \rangle + \frac{1}{2} \langle \eta, \operatorname{Hess} f(x_k) \eta \rangle + h(x_k + \eta) \\ x_{k+1} = R_{x_k}(\eta_k) \end{cases}$$

- $x_k + \eta$ in h is only a first order approximation;

A Riemannian Proximal Newton-CG Method

A native generalization

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- $x_k + \eta$ in h is only a first order approximation;
- If a second order approximation is used, then the subproblem is difficult to solve;

A Riemannian proximal Newton-CG method

A Riemannian proximal Newton method: description

$$\min_{x \in \mathcal{M}} F(x) = f(x) + h(x), h(x) = \mu \|x\|_1$$

A Riemannian proximal Newton method (RPN)

- 1 Compute the ManPG direction

$$v(x_k) = \operatorname{argmin}_{v \in T_{x_k} \mathcal{M}} f(x_k) + \langle \nabla f(x_k), v \rangle + \frac{1}{2t} \|v\|_F^2 + h(x_k + v);$$

- 2 Find $u(x_k) \in T_{x_k} \mathcal{M}$ by solving

$$J(x_k)[u(x_k)] = -v(x_k),$$

where $J(x_k) = -[I_n - \Lambda_{x_k} + t\Lambda_{x_k}(\nabla^2 f(x_k) - \mathcal{L}_{x_k})]$, Λ_{x_k} and \mathcal{L}_{x_k} are defined later;

- 3 $x_{k+1} = R_{x_k}(u(x_k));$

A Riemannian proximal Newton-CG method

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- 1 Step 1: compute a Riemannian proximal gradient direction (ManPG)

A Riemannian proximal Newton-CG method

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- 3 $x_{k+1} = R_{x_k}(u(x_k))$;

- 1 Step 1: compute a Riemannian proximal gradient direction (ManPG)
- 2 Step 2: compute the Riemannian proximal Newton direction, where $J(x_k)$ is from a generalized Jacobi of $v(x_k)$;

A Riemannian proximal Newton-CG method

A Riemannian proximal Newton method: description

$$\min_{x \in \mathcal{M}} F(x) = f(x) + h(x), h(x) = \mu \|x\|_1$$

A Riemannian proximal Newton method (RPN)

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$$v(x_k) = \operatorname{argmin}_{v \in T_{x_k} \mathcal{M}} f(x_k) + \langle \nabla f(x_k), v \rangle + \frac{1}{2t} \|v\|_F^2 + h(x_k + v);$$

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- 1 Step 1: compute a Riemannian proximal gradient direction (ManPG)
- 2 Step 2: compute the Riemannian proximal Newton direction, where $J(x_k)$ is from a generalized Jacobi of $v(x_k)$;
- 3 Step 3: Update iterate by a retraction;

A Riemannian proximal Newton-CG method

A Riemannian proximal Newton method: local superlinear convergence rate

Without loss of generality, we assume that the nonzero entries of x_* are in the first part, i.e., $x_* = [\bar{x}_*^T, 0^T]^T$. B_x denotes an orthonormal basis of $T_x^\perp \mathcal{M}$ at x .

Assumption:

- Let $B_{x_*}^T = [\bar{B}_{x_*}^T, \hat{B}_{x_*}^T]$, where $\bar{B}_{x_*} \in \mathbb{R}^{j \times d}$ and $\hat{B}_{x_*} \in \mathbb{R}^{(n-j) \times d}$. It is assumed that $j \geq d$ and \bar{B}_{x_*} is full column rank;

A Riemannian proximal Newton-CG method

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- 2 There exists a neighborhood \mathcal{U} of $x_* = [\bar{x}_*^T, 0^T]^T$ on \mathcal{M} such that for any $x = [\bar{x}^T, \hat{x}^T]^T \in \mathcal{U}$, it holds that $\bar{x} + \bar{v} \neq 0$ and $\hat{x} + \hat{v} = 0$.

A Riemannian proximal Newton-CG method

A Riemannian proximal Newton method: local superlinear convergence rate

Theorem

Suppose that x_ be a local optimal minimizer. Under the above Assumptions, assume that $J(x_*)$ is nonsingular. Then there exists a neighborhood \mathcal{U} of x_* on \mathcal{M} such that for any $x_0 \in \mathcal{U}$, RPN Algorithm generates the sequence $\{x_k\}$ converging superlinearly to x_* .*

The convergence rate is improved to quadratically convergence in [SAH⁺24a]

A Riemannian proximal Newton-CG method

A Riemannian proximal Newton method: a hybrid version

- Similar to the Riemannian Newton method, this Riemannian proximal Newton method does not guarantee global convergence;

A Riemannian proximal Newton-CG method

A Riemannian proximal Newton method: a hybrid version

- Similar to the Riemannian Newton method, this Riemannian proximal Newton method does not guarantee global convergence;
- A hybrid method that merges ManPG with RPN is proposed in [SAH⁺24b];

Require: $x_0 \in \mathcal{M}$, $t > 0$, $\epsilon > 0$;

- 1: **for** $k = 0, 1, \dots$ **do**
 - 2: Compute a ManPG direction v_k ;
 - 3: If $\|v_k\| \leq \epsilon$, then $K = k$ and break;
 - 4: $x_{k+1} = R_{x_k}(\alpha v_k)$ with an appropriate step size;
 - 5: **end for**
 - 6: **for** $k = K+1, K+2, \dots$ **do**
 - 7: Compute u_k by solving $J(x_k)u_k = -v_k$ with v_k being the ManPG direction;
 - 8: $x_{k+1} = R_{x_k}(u_k)$;
 - 9: **end for**
-

A Riemannian proximal Newton-CG method

A Riemannian proximal Newton method: a hybrid version

- Similar to the Riemannian Newton method, this Riemannian proximal Newton method does not guarantee global convergence;
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- 8: $x_{k+1} = R_{x_k}(u_k)$;
- 9: **end for**

The switching parameter ϵ is crucial for the performance.

A Riemannian proximal Newton-CG method

Truncated conjugate gradient

A Riemannian proximal Newton method (RPN)

- 1 Compute the ManPG direction

$$v(x_k) = \operatorname{argmin}_{v \in T_{x_k} \mathcal{M}} f(x_k) + \langle \nabla f(x_k), v \rangle + \frac{1}{2t} \|v\|_F^2 + h(x_k + v);$$

- 2 Find $u(x_k) \in T_{x_k} \mathcal{M}$ by solving

$$J(x_k)[u(x_k)] = -v(x_k);$$

- 3 $x_{k+1} = R_{x_k}(u(x_k));$

Smooth case:

- $v(x_k) = -t \operatorname{grad} f(x_k);$
- $J(x_k) = -t \operatorname{Hess} f(x_k);$
- $J(x_k)[u(x_k)] = -v(x_k) \implies$
 $\underbrace{\operatorname{Hess} f(x_k)[u(x_k)]}_{\text{truncated conjugate gradient (tCG)}} = -\operatorname{grad} f(x_k) .$

A Riemannian proximal Newton-CG method

Truncated conjugate gradient

A Riemannian proximal Newton method (RPN)

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$$v(x_k) = \operatorname{argmin}_{v \in T_{x_k} \mathcal{M}} f(x_k) + \langle \nabla f(x_k), v \rangle + \frac{1}{2t} \|v\|_F^2 + h(x_k + v);$$

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Nonsmooth case:

- $v(x_k)$: ManPG direction;
- $J(x_k)$: Generalized Jacobi of v ;
- $u(x_k)$: solving a linear system by
 $\underbrace{J(x_k)[u(x_k)] = -v(x_k)}_{\text{tCG?}}$

A Riemannian proximal Newton-CG method

Truncated conjugate gradient

A Riemannian proximal Newton method (RPN)

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- 2 Find $u(x_k) \in T_{x_k} \mathcal{M}$ by solving

$$J(x_k)[u(x_k)] = -v(x_k);$$

- 3 $x_{k+1} = R_{x_k}(u(x_k));$

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Problem: $J(x_k)$ is not symmetric!

A Riemannian proximal Newton-CG method

Truncated conjugate gradient

Notation:

$$\mathfrak{B}_{x_k} = \nabla^2 f(x_k) - \mathcal{L}_{x_k} = \begin{pmatrix} \mathfrak{B}_{x_k}^{(11)} & \mathfrak{B}_{x_k}^{(12)} \\ \mathfrak{B}_{x_k}^{(21)} & \mathfrak{B}_{x_k}^{(22)} \end{pmatrix}, \mathcal{B}_{x_k} = \mathfrak{B}_{x_k}^{(11)}.$$

$$J(x_k) = - \begin{pmatrix} \bar{B}_{x_k} \bar{B}_{x_k}^\dagger + t(l_{j_k} - \bar{B}_{x_k} \bar{B}_{x_k}^\dagger) \mathcal{B}_{x_k} & t(l_{j_k} - \bar{B}_{x_k} \bar{B}_{x_k}^\dagger) \mathfrak{B}_{x_k}^{(12)} \\ 0_{(n-j_k) \times j_k} & l_{n-j_k} \end{pmatrix}$$

$$\begin{cases} [\bar{B}_{x_k} \bar{B}_{x_k}^\dagger + t(l_{j_k} - \bar{B}_{x_k} \bar{B}_{x_k}^\dagger) \mathcal{B}_{x_k}] \bar{u}(x_k) = \bar{v}(x_k) - t(l_{j_k} - \bar{B}_{x_k} \bar{B}_{x_k}^\dagger) \mathfrak{B}_{x_k}^{(12)} \hat{u}(x_k) \\ \hat{u}(x_k) = \hat{v}(x_k) \end{cases}.$$

$$\implies \bar{u}(x_k) = \bar{v}(x_k) - \{l_{j_k} + (l_{j_k} - \bar{B}_{x_k} \bar{B}_{x_k}^\dagger) N_{x_k}\}^{-1} (l_{j_k} - \bar{B}_{x_k} \bar{B}_{x_k}^\dagger) \ell_{x_k}$$

where $\ell_{x_k} = \frac{1}{t_k} (-l_{j_k} + t_k \mathcal{B}_{x_k}) \bar{v}(x_k) + \mathfrak{B}_{x_k}^{(12)} \hat{v}(x_k)$ and $N_{x_k} = -l_{j_k} + t \mathcal{B}_{x_k}$ is symmetric.

A Riemannian proximal Newton-CG method

Truncated conjugate gradient

$$\bar{u}(x_k) = \bar{v}(x_k) - \{I_{j_k} + (I_{j_k} - \bar{B}_{x_k} \bar{B}_{x_k}^\dagger) \underbrace{N_{x_k}}_{\text{symmetric}}\}^{-1} (I_{j_k} - \bar{B}_{x_k} \bar{B}_{x_k}^\dagger) \ell_{x_k}$$

Lemma

Let $N \in \mathbb{R}^{j \times j}$ and $B \in \mathbb{R}^{j \times m}$ with $m \leq j$. Suppose that $I_j + N$ is symmetric positive definite on $\{w \mid B^T w = 0\}$ and that B is full column rank. Then it holds that the unique solution of the problem

$$\min_{B^T w = 0} \ell^T w + \frac{1}{2} w^T (I_j + N) w$$

is given by

$$w_* = - [I_j + (I_j - BB^\dagger)N]^{-1} [I_j - BB^\dagger] \ell.$$

A Riemannian proximal Newton-CG method

Truncated conjugate gradient

$$\bar{u}(x_k) = \bar{v}(x_k) - \{I_{j_k} + (I_{j_k} - \bar{B}_{x_k} \bar{B}_{x_k}^\dagger) \underbrace{N_{x_k}}_{\text{symmetric}}\}^{-1} (I_{j_k} - \bar{B}_{x_k} \bar{B}_{x_k}^\dagger) \ell_{x_k}$$

Corollary

Suppose \bar{B}_{x_k} has full column rank, B_{x_k} is symmetric positive definite on $\{w \mid B^T w = 0\}$. Then the proximal Newton equation

$J(x_k)[u(x_k)] = -v(x_k)$ can be computed by

$$u(x_k) = \begin{pmatrix} \bar{v}(x_k) + w(x_k) \\ \hat{v}(x_k) \end{pmatrix},$$

where $w(x_k) = \operatorname{argmin}_{\bar{B}_{x_k}^T w = 0} \ell_{x_k}^T w + \frac{1}{2} w^T B_{x_k} w$.

A Riemannian proximal Newton-CG method

Truncated conjugate gradient

$$\bar{u}(x_k) = \bar{v}(x_k) - \{I_{j_k} + (I_{j_k} - \bar{B}_{x_k} \bar{B}_{x_k}^\dagger) \underbrace{N_{x_k}}_{\text{symmetric}}\}^{-1} (I_{j_k} - \bar{B}_{x_k} \bar{B}_{x_k}^\dagger) \ell_{x_k}$$

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Suppose \bar{B}_{x_k} has full column rank, B_{x_k} is symmetric positive definite on $\{w \mid B^T w = 0\}$. Then the proximal Newton equation

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tCG can be used for the computation of $w(x_k)$.

A Riemannian proximal Newton-CG method

Truncated conjugate gradient

A Riemannian proximal Newton method (RPN)

- 1 $v(x_k) = \operatorname{argmin}_{v \in T_{x_k} \mathcal{M}} f(x_k) + \langle \nabla f(x_k), v \rangle + \frac{1}{2t} \|v\|_F^2 + h(x_k + v);$
- 2 $d(x_k) = \begin{pmatrix} \bar{d}(x_k) \\ \hat{d}(x_k) \end{pmatrix} = \begin{pmatrix} \bar{v}(x_k) + w(x_k) \\ \hat{v}(x_k) \end{pmatrix}$, where $w(x_k)$ is an output of tCG for solving $\min_{\bar{B}_{x_k}^T w=0} \langle \ell_{x_k}, w \rangle + \frac{1}{2} \langle w, \mathcal{B}_{x_k} w \rangle$.
- 3 $x_{k+1} = R_{x_k}(\alpha_k d(x_k))$ with an appropriate step size α_k ;

Question:

- Is \mathcal{B}_{x_k} symmetric positive definite near a local minimizer x_* ?
- What is the early termination conditions for tCG?
 - Guarantee global convergence;
 - Guarantee local superlinear convergence;

A Riemannian proximal Newton-CG method

Truncated conjugate gradient

Is \mathcal{B}_{x_k} symmetric positive definite near x_* ?

A Riemannian proximal Newton-CG method

Truncated conjugate gradient

Is \mathcal{B}_{x_k} symmetric positive definite near x_* ?

Assumption:

- 1 The function f is twice continuously differentiable with a Lipschitz continuous Euclidean Hessian;
- 2 Let $B_{x_*}^T = [\bar{B}_{x_*}^T, \hat{B}_{x_*}^T]$, where $\bar{B}_{x_*} \in \mathbb{R}^{j \times d}$ and $\hat{B}_{x_*} \in \mathbb{R}^{(n-j) \times d}$. It is assumed that $j \geq d$ and \bar{B}_{x_*} is full column rank;
- 3 There exists a neighborhood \mathcal{U} of $x_* = [\bar{x}_*^T, 0^T]^T$ on \mathcal{M} such that for any $x = [\bar{x}^T, \tilde{x}^T]^T \in \mathcal{U}$, it holds that $\bar{x} + \bar{v} \neq 0$ and $\hat{x} + \hat{v} = 0$;
- 4 The linear operator \mathcal{B}_{x_*} is positive definite on the subspace $\mathfrak{L}_{x_*} = \{w \mid \bar{B}_{x_*}^T w = 0\}$.

A Riemannian proximal Newton-CG method

Truncated conjugate gradient

Is \mathcal{B}_{x_k} symmetric positive definite near x_* ?

Assumption:

- 1 The function f is twice continuously differentiable with a Lipschitz continuous Euclidean Hessian;
 - 2 Let $B_{x_*}^T = [\bar{B}_{x_*}^T, \hat{B}_{x_*}^T]$, where $\bar{B}_{x_*} \in \mathbb{R}^{j \times d}$ and $\hat{B}_{x_*} \in \mathbb{R}^{(n-j) \times d}$. It is assumed that $j \geq d$ and \bar{B}_{x_*} is full column rank;
 - 3 There exists a neighborhood \mathcal{U} of $x_* = [\bar{x}_*^T, 0^T]^T$ on \mathcal{M} such that for any $x = [\bar{x}^T, \tilde{x}^T]^T \in \mathcal{U}$, it holds that $\bar{x} + \bar{v} \neq 0$ and $\hat{x} + \hat{v} = 0$;
 - 4 The linear operator \mathcal{B}_{x_*} is positive definite on the subspace $\mathcal{L}_{x_*} = \{w \mid \bar{B}_{x_*}^T w = 0\}$.
-

- Under the second assumption, the intersection of the manifold and the sparsity constraints forms an embedded submanifold around x_* ;
- \mathcal{B}_{x_*} is the Riemannian Hessian of F at x_* for the submanifold;
- \mathcal{B}_{x_*} is symmetric positive semidefinite on \mathcal{L}_{x_*} ;

A Riemannian proximal Newton-CG method

Truncated conjugate gradient

Is \mathcal{B}_{x_k} symmetric positive definite near x_* ?

Assumption:

- 1 The function f is twice continuously differentiable with a Lipschitz continuous Euclidean Hessian;
- 2 Let $B_{x_*}^T = [\bar{B}_{x_*}^T, \hat{B}_{x_*}^T]$, where $\bar{B}_{x_*} \in \mathbb{R}^{j \times d}$ and $\hat{B}_{x_*} \in \mathbb{R}^{(n-j) \times d}$. It is assumed that $j \geq d$ and \bar{B}_{x_*} is full column rank;
- 3 There exists a neighborhood \mathcal{U} of $x_* = [\bar{x}_*^T, 0^T]^T$ on \mathcal{M} such that for any $x = [\bar{x}^T, \tilde{x}^T]^T \in \mathcal{U}$, it holds that $\bar{x} + \bar{v} \neq 0$ and $\hat{x} + \hat{v} = 0$;
- 4 The linear operator \mathcal{B}_{x_*} is positive definite on the subspace $\mathfrak{L}_{x_*} = \{w \mid \bar{B}_{x_*}^T w = 0\}$.

Lemma

Suppose the above Assumption holds. Then there exists a neighborhood of x_* , denoted by \mathcal{V}_2 , and a positive constant χ_ϵ such that the smallest eigenvalue of \mathcal{B}_x on \mathfrak{L}_x is greater than χ_ϵ for all $x \in \mathcal{V}_2$. This implies \mathcal{B}_x is positive definite on \mathfrak{L}_x for all $x \in \mathcal{V}_2$.

A Riemannian proximal Newton-CG method

Truncated conjugate gradient

Early termination conditions in tCG

tCG step

- 2 $d(x_k) = \begin{pmatrix} \bar{d}(x_k) \\ \hat{d}(x_k) \end{pmatrix} = \begin{pmatrix} \bar{v}(x_k) + w(x_k) \\ \hat{v}(x_k) \end{pmatrix}$, where $w(x_k)$ is an output of tCG for solving $\min_{\bar{B}_{x_k}^T w=0} \langle \ell_{x_k}, w \rangle + \frac{1}{2} \langle w, \mathcal{B}_{x_k} w \rangle$.

A Riemannian proximal Newton-CG method

Truncated conjugate gradient

Early termination conditions in tCG

tCG step

- 2 $d(x_k) = \begin{pmatrix} \bar{d}(x_k) \\ \hat{d}(x_k) \end{pmatrix} = \begin{pmatrix} \bar{v}(x_k) + w(x_k) \\ \hat{v}(x_k) \end{pmatrix}$, where $w(x_k)$ is an output of tCG for solving $\min_{\bar{B}_{x_k}^T w=0} \langle \ell_{x_k}, w \rangle + \frac{1}{2} \langle w, \mathcal{B}_{x_k} w \rangle$.

Difficulty

- Smooth:

$$\text{approximately } \min_{d \in T_{x_k} \mathcal{M}} \langle \text{grad } f(x_k), d \rangle + \frac{1}{2} \langle \text{Hess } f(x_k)[d], d \rangle,$$

find $d(x_k)$ such that $\langle d(x_k), \text{grad } f(x_k) \rangle < 0$;

- Nonsmooth:

$$\text{approximately } \min_{\bar{B}_{x_k}^T w=0} \langle \ell_{x_k}, w \rangle + \frac{1}{2} \langle w, \mathcal{B}_{x_k} w \rangle,$$

find $w(x_k)$ such that $d(x_k)$ is a descent direction;

A Riemannian proximal Newton-CG method

Truncated conjugate gradient

Early termination conditions in tCG

tCG step

- 2 $d(x_k) = \begin{pmatrix} \bar{d}(x_k) \\ \hat{d}(x_k) \end{pmatrix} = \begin{pmatrix} \bar{v}(x_k) + w(x_k) \\ \hat{v}(x_k) \end{pmatrix}$, where $w(x_k)$ is an output of tCG for solving $\min_{\bar{B}_{x_k}^T w=0} \langle \ell_{x_k}, w \rangle + \frac{1}{2} \langle w, \mathcal{B}_{x_k} w \rangle$.

Difficulty

- Smooth:

$$\text{approximately } \min_{d \in T_{x_k} \mathcal{M}} \langle \text{grad } f(x_k), d \rangle + \frac{1}{2} \langle \text{Hess } f(x_k)[d], d \rangle,$$

find $d(x_k)$ such that $\langle d(x_k), \text{grad } f(x_k) \rangle < 0$;

- Nonsmooth:

$$\text{approximately } \min_{\bar{B}_{x_k}^T w=0} \langle \ell_{x_k}, w \rangle + \frac{1}{2} \langle w, \mathcal{B}_{x_k} w \rangle,$$

find $w(x_k)$ such that $d(x_k)$ is a descent direction;

The early termination conditions for the smooth case are not sufficient.

A Riemannian proximal Newton-CG method

Truncated conjugate gradient

Early termination conditions in tCG

Algorithm: Truncated conjugate gradient (tCG)

Require: $\vartheta > 0$, $\gamma > 0$, $\tau > 0$, $\theta > 0$, and $\kappa \in (0, 1)$;

Ensure: $(w(x), \text{status})$;

1: **if** $G_x(v(x)) > G_x(0)$ **then**

2: return $w(x) = 0$ and $\text{status} = \text{'early1'}$;

3: **end if**

4: $z = \mathfrak{B}v(x)$;

5: **if** $\langle v(x), z \rangle + \tau \|\hat{v}(x)\|_F^2 < \gamma \|v(x)\|_F^2$ **then**

6: return $w(x) = 0$ and $\text{status} = \text{'early2'}$;

7: **end if**

8: $w_0 = 0$, $r_0 = P_x(\ell_x)$, $\alpha_0 = -r_0$, $\delta_0 = \langle r_0, r_0 \rangle$, $t_0 = z$;

9: (CG iterations)

Omit subscript k for simplicity

A Riemannian proximal Newton-CG method

Truncated conjugate gradient

Early termination conditions in tCG

Algorithm: Truncated conjugate gradient (tCG)

Require: $\vartheta > 0$, $\gamma > 0$, $\tau > 0$, $\theta > 0$, and $\kappa \in (0, 1)$;

Ensure: $(w(x), \text{status})$;

1: **if** $G_x(v(x)) > G_x(0)$ **then**

2: return $w(x) = 0$ and status = 'early1';

3: **end if**

4: $z = \mathfrak{B}v(x)$;

5: **if** $\langle v(x), z \rangle + \tau \|\hat{v}(x)\|_F^2 < \gamma \|v(x)\|_F^2$ **then**

6: return $w(x) = 0$ and status = 'early2';

7: **end if**

8: $w_0 = 0$, $r_0 = P_x(\ell_x)$, $\alpha_0 = -r_0$, $\delta_0 = \langle r_0, r_0 \rangle$, $t_0 = z$;

9: (CG iterations)

- $G_x(u) = f(x) + \langle \nabla f(x), u \rangle + \frac{1}{2} \langle u, \mathfrak{B}_x u \rangle + \frac{\tau}{2} \|\hat{u}(x)\|_F^2 + h(x + u)$;
- Use to guarantee global convergence;
- $\frac{\tau}{2} \|\hat{u}(x)\|_F^2$ is added for the condition in Step 5;

A Riemannian proximal Newton-CG method

Truncated conjugate gradient

Early termination conditions in tCG

Algorithm: Truncated conjugate gradient (tCG)

Require: $\vartheta > 0$, $\gamma > 0$, $\tau > 0$, $\theta > 0$, and $\kappa \in (0, 1)$;

Ensure: $(w(x), \text{status})$;

1: **if** $G_x(v(x)) > G_x(0)$ **then**

2: return $w(x) = 0$ and status = 'early1';

3: **end if**

4: $z = \mathfrak{B}v(x)$;

5: **if** $\langle v(x), z \rangle + \tau \|\hat{v}(x)\|_F^2 < \gamma \|v(x)\|_F^2$ **then**

6: return $w(x) = 0$ and status = 'early2';

7: **end if**

8: $w_0 = 0$, $r_0 = P_x(\ell_x)$, $\alpha_0 = -r_0$, $\delta_0 = \langle r_0, r_0 \rangle$, $t_0 = z$;

9: (CG iterations)

- Use to guarantee global convergence;
- $\tau \|\hat{v}(x)\|_F^2$ is used since $\mathfrak{B}_x \succ 0$ may not hold;

A Riemannian proximal Newton-CG method

Truncated conjugate gradient

Early termination conditions in tCG

Algorithm: Truncated conjugate gradient (tCG)

Require: $\vartheta > 0$, $\gamma > 0$, $\tau > 0$, $\theta > 0$, and $\kappa \in (0, 1)$;

Ensure: $(w(x), \text{status})$;

- 1: (See the previous slide)
 - 2: $w_0 = 0$, $r_0 = P_x(\ell_x)$, $o_0 = -r_0$, $\delta_0 = \langle r_0, r_0 \rangle$, $t_0 = z$;
 - 3: **for** $i = 0, 1, \dots$ **do**
 - 4: $p_i = \mathcal{B}o_i$ and $q_i = P_x(p_i)$;
 - 5: **if** $\langle o_i, q_i \rangle \leq \vartheta \delta_i$ **then**
 - 6: return $w(x) = w_i$ and status = 'neg';
 - 7: **end if**
 - 8: (Remaining CG iterations)
 - 9: **end for**
-

An existing early termination condition

A Riemannian proximal Newton-CG method

Truncated conjugate gradient

Early termination conditions in tCG

Algorithm: Truncated conjugate gradient (tCG)

Require: $\vartheta > 0$, $\gamma > 0$, $\tau > 0$, $\theta > 0$, and $\kappa \in (0, 1)$;

Ensure: $(w(x), \text{status})$;

- 1: (See previous slides)
 - 2: **for** $i = 0, 1, \dots$ **do**
 - 3: (See previous slides)
 - 4: $\alpha_i = \frac{\langle r_i, r_i \rangle}{\langle o_i, q_i \rangle}$; $w_{i+1} = w_i + \alpha_i o_i$; $r_{i+1} = r_i + \alpha_i q_i$;
 - 5: $d_{i+1} = \begin{pmatrix} \bar{v}(x) + w_{i+1} \\ \hat{v}(x) \end{pmatrix}$, $t_{i+1} = t_i + \alpha_i \begin{pmatrix} p_i \\ \mathfrak{B}_{21} o_i \end{pmatrix}$;
 - 6: **if** $\langle d_{i+1}, t_{i+1} \rangle + \tau \|\hat{v}(x)\|_F^2 < \gamma \|d_{i+1}\|_F^2$ or $G_x(d_{i+1}) > G_x(0)$ **then**
 - 7: **return** $w(x) = w_i$ and **status** = 'early3';
 - 8: **end if**
 - 9: (Remaining CG iterations)
 - 10: **end for**
-

Use to guarantee global convergence

A Riemannian proximal Newton-CG method

Truncated conjugate gradient

Early termination conditions in tCG

Algorithm: Truncated conjugate gradient (tCG)

Require: $\vartheta > 0$, $\gamma > 0$, $\tau > 0$, $\theta > 0$, and $\kappa \in (0, 1)$;

Ensure: $(w(x), \text{status})$;

- 1: (See previous slides)
 - 2: **for** $i = 0, 1, \dots$ **do**
 - 3: (See previous slides)
 - 4: $\beta_{i+1} = \frac{\langle r_{i+1}, r_{i+1} \rangle}{\langle r_i, r_i \rangle}$; $o_{i+1} = -r_{i+1} + \beta_{i+1} o_i$;
 - 5: $\delta_{i+1} = \langle r_{i+1}, r_{i+1} \rangle + \beta_{i+1}^2 \delta_i$; (Note that $\delta_{i+1} = \langle o_{i+1}, o_{i+1} \rangle$)
 - 6: $i = i + 1$;
 - 7: **if** $\|r_i\|_F \leq \|r_0\|_F \min(\|r_0\|_F^\theta, \kappa)$ **then**
 - 8: **return** $w(x) = w_i$, and **status** = 'lin' if $\|r_0\|_F^\theta > \kappa$ and **status** = 'sup' otherwise;
 - 9: **end if**
 - 10: **end for**
-

An existing early termination condition

A Riemannian proximal Newton-CG method

RPN-CG: global convergence

Assumption:

- 1 The function f is twice continuously differentiable with a Lipschitz continuous gradient;

Theorem

Suppose the above Assumption holds and the parameters are appropriately chosen. Then it holds that

$$\lim_{k \rightarrow \infty} \|v(x_k)\|_F = 0.$$

A Riemannian proximal Newton-CG method

RPN-CG: local superlinear convergence

Assumption:

- 1 The function f is twice continuously differentiable with a Lipschitz continuous Euclidean Hessian;
- 2 Let $B_{x_*}^T = [\bar{B}_{x_*}^T, \hat{B}_{x_*}^T]$, where $\bar{B}_{x_*} \in \mathbb{R}^{j \times d}$ and $\hat{B}_{x_*} \in \mathbb{R}^{(n-j) \times d}$. It is assumed that $j \geq d$ and \bar{B}_{x_*} is full column rank;
- 3 There exists a neighborhood \mathcal{U} of $x_* = [\bar{x}_*^T, 0^T]^T$ on \mathcal{M} such that for any $x = [\bar{x}^T, \tilde{x}^T]^T \in \mathcal{U}$, it holds that $\bar{x} + \bar{v} \neq 0$ and $\hat{x} + \hat{v} = 0$;
- 4 The function F is ζ -geodesically strongly convex at x_* , i.e., there exists a neighborhood $\tilde{\mathcal{U}}_{x_*}$ of x_* in \mathcal{M} such that

$$F(y) \geq F(x_*) + \frac{\zeta}{2} \|\text{Exp}_{x_*}^{-1}(y)\|_F^2$$

holds for any $y \in \tilde{\mathcal{U}}_{x_*}$.

A Riemannian proximal Newton-CG method

RPN-CG: local superlinear convergence

Assumption:

- 1 The function f is twice continuously differentiable with a Lipschitz continuous Euclidean Hessian;
- 2 Let $B_{x_*}^T = [\bar{B}_{x_*}^T, \hat{B}_{x_*}^T]$, where $\bar{B}_{x_*} \in \mathbb{R}^{j \times d}$ and $\hat{B}_{x_*} \in \mathbb{R}^{(n-j) \times d}$. It is assumed that $j \geq d$ and \bar{B}_{x_*} is full column rank;
- 3 There exists a neighborhood \mathcal{U} of $x_* = [\bar{x}_*^T, 0^T]^T$ on \mathcal{M} such that for any $x = [\bar{x}^T, \tilde{x}^T]^T \in \mathcal{U}$, it holds that $\bar{x} + \bar{v} \neq 0$ and $\hat{x} + \hat{v} = 0$;
- 4 The function F is ζ -geodesically strongly convex at x_* , i.e., there exists a neighborhood $\tilde{\mathcal{U}}_{x_*}$ of x_* in \mathcal{M} such that

$$F(y) \geq F(x_*) + \frac{\zeta}{2} \|\text{Exp}_{x_*}^{-1}(y)\|_F^2$$

holds for any $y \in \tilde{\mathcal{U}}_{x_*}$.

Lemma

Suppose the last Assumption holds, that is, the function $F = f + h$ is ζ -geodesically strongly convex at x_* . Then the linear operator \mathcal{B}_{x_*} is positive definite on \mathfrak{L}_{x_*} .

A Riemannian proximal Newton-CG method

RPN-CG: local superlinear convergence

Assumption:

- 1 The function f is twice continuously differentiable with a Lipschitz continuous Euclidean Hessian;
- 2 Let $B_{x_*}^T = [\bar{B}_{x_*}^T, \hat{B}_{x_*}^T]$, where $\bar{B}_{x_*} \in \mathbb{R}^{j \times d}$ and $\hat{B}_{x_*} \in \mathbb{R}^{(n-j) \times d}$. It is assumed that $j \geq d$ and \bar{B}_{x_*} is full column rank;
- 3 There exists a neighborhood \mathcal{U} of $x_* = [\bar{x}_*^T, 0^T]^T$ on \mathcal{M} such that for any $x = [\bar{x}^T, \tilde{x}^T]^T \in \mathcal{U}$, it holds that $\bar{x} + \bar{v} \neq 0$ and $\hat{x} + \hat{v} = 0$;
- 4 The function F is ζ -geodesically strongly convex at x_* , i.e., there exists a neighborhood $\tilde{\mathcal{U}}_{x_*}$ of x_* in \mathcal{M} such that

$$F(y) \geq F(x_*) + \frac{\zeta}{2} \|\text{Exp}_{x_*}^{-1}(y)\|_F^2$$

holds for any $y \in \tilde{\mathcal{U}}_{x_*}$.

Theorem

Suppose the previous assumptions hold. If x is sufficiently close x_ and the parameters are appropriately chosen, then tCG terminates only due to the accurate condition, i.e., $\|r_i\|_F \leq \|r_0\|_F \min(\|r_0\|_F^\theta, \kappa)$.*

A Riemannian proximal Newton-CG method

RPN-CG: local superlinear convergence

Theorem

Suppose the previous Assumptions hold and the parameters are appropriately chosen. Then there exists a neighborhood of x_ , denoted by \mathcal{V}_δ , such that if the step size one is used, then the convergence rate is $\min(1 + \theta, 2)$, i.e., $\|R_x(d(x)) - x_*\|_F \leq C_{\text{up}} \|x - x_*\|_F^{\min(1+\theta, 2)}$ holds for any $x \in \mathcal{V}_\delta$ and a constant $C_{\text{up}} > 0$.*

A Riemannian proximal Newton-CG method

RPN-CG: local superlinear convergence

Theorem

Suppose the previous Assumptions hold and the parameters are appropriately chosen. Then there exists a neighborhood of x_ , denoted by \mathcal{V}_δ , such that if the step size one is used, then the convergence rate is $\min(1 + \theta, 2)$, i.e., $\|R_x(d(x)) - x_*\|_F \leq C_{\text{up}} \|x - x_*\|_F^{\min(1+\theta, 2)}$ holds for any $x \in \mathcal{V}_\delta$ and a constant $C_{\text{up}} > 0$.*

Is step size one acceptable for x sufficiently close to x_* ?
That is to make objective function sufficiently descent.

A Riemannian proximal Newton-CG method

RPN-CG: local superlinear convergence

Theorem

Suppose the previous Assumptions hold and the parameters are appropriately chosen. Then there exists a neighborhood of x_ , denoted by \mathcal{V}_8 , such that if the step size one is used, then the convergence rate is $\min(1 + \theta, 2)$, i.e., $\|R_x(d(x)) - x_*\|_F \leq C_{\text{up}} \|x - x_*\|_F^{\min(1+\theta, 2)}$ holds for any $x \in \mathcal{V}_8$ and a constant $C_{\text{up}} > 0$.*

Is step size one acceptable for x sufficiently close to x_* ?

That is to make objective function sufficiently descent.

- For smooth Riemannian optimization problem, step size one is acceptable eventually for Riemannian Newton method;
- For Euclidean nonsmooth optimization problem $F = f + g$, step size one is also acceptable eventually for proximal Newton method [LSS14];

A Riemannian proximal Newton-CG method

RPN-CG: local superlinear convergence

Example

- Consider $F : \mathbb{R}^2 \rightarrow \mathbb{R} : (x_1, x_2)^T \mapsto \underbrace{x_1^2 - 3x_1 + 1 + x_2^2}_{f(x)} + \underbrace{|x_1| + |x_2|}_{g(x)}$;
- The unique minimizer: $x_* = (1, 0)^T$;
- $x = (1 + \epsilon, 0)^T$ with $|\epsilon|$ being arbitrarily small;
- Proximal Newton direction: $u(x) = -(\epsilon, 0)^T$;
- Retraction: $R : \mathbb{T}\mathcal{M} \rightarrow \mathcal{M} : \eta_x \mapsto x + \eta_x + \begin{pmatrix} 0 \\ 2\eta_x^T \eta_x \end{pmatrix}$;
- $R(u(x)) = (1, 2\epsilon^2)^T$;
- $F(R_x(u(x))) - F(x) = 4\epsilon^4 + \epsilon^2 > 0$;
- Step size one is not acceptable for any $\epsilon > 0$;

A Riemannian proximal Newton-CG method

RPN-CG: local superlinear convergence

Example

- Consider $F : \mathbb{R}^2 \rightarrow \mathbb{R} : (x_1, x_2)^T \mapsto \underbrace{x_1^2 - 3x_1 + 1 + x_2^2}_{f(x)} + \underbrace{|x_1| + |x_2|}_{g(x)}$;
- The unique minimizer: $x_* = (1, 0)^T$;
- $x = (1 + \epsilon, 0)^T$ with $|\epsilon|$ being arbitrarily small;
- Proximal Newton direction: $u(x) = -(\epsilon, 0)^T$;
- Retraction: $R : \mathbb{T}\mathcal{M} \rightarrow \mathcal{M} : \eta_x \mapsto x + \eta_x + \begin{pmatrix} 0 \\ 2\eta_x^T \eta_x \end{pmatrix}$;
- $R(u(x)) = (1, 2\epsilon^2)^T$;
- $F(R_x(u(x))) - F(x) = 4\epsilon^4 + \epsilon^2 > 0$;
- Step size one is not acceptable for any $\epsilon > 0$;

The answer is negative for nonsmooth Riemannian problems.

Difficulty comes from the nonsmoothness and the curvature.

A Riemannian proximal Newton-CG method

RPN-CG: local superlinear convergence

Two consecutive iterations near x_* guarantee sufficient descent.

Theorem

Suppose that the previous Assumptions hold and that there exists a neighborhood of x_ , denoted by \mathcal{V}_9 , such that for any $x \in \mathcal{V}_9$, it holds that $\|R_x(d(x)) - x_*\|_F \leq C_{\text{up}}\|x - x_*\|_F^\varkappa$ for a $\varkappa > \sqrt{2}$ and $R_x(d(x)) \in \mathcal{V}_9$. Then there exists a neighborhood of x_* , denoted by \mathcal{V}_{10} , and a constant $\rho_1 > 0$ such that for any $x \in \mathcal{V}_{10}$, it holds that*

$$F(x_{++}) \leq F(x) - \rho_1 \|v(x)\|_F^2,$$

where $x_+ = R_x(d(x))$ and $x_{++} = R_{x_+}(d(x_+))$.

A Riemannian proximal Newton-CG method

RPN-CG: local superlinear convergence

Two consecutive iterations near x_* guarantee sufficient descent.

Theorem

Suppose that the previous Assumptions hold and that there exists a neighborhood of x_ , denoted by \mathcal{V}_9 , such that for any $x \in \mathcal{V}_9$, it holds that $\|R_x(d(x)) - x_*\|_F \leq C_{\text{up}}\|x - x_*\|_F^\varkappa$ for a $\varkappa > \sqrt{2}$ and $R_x(d(x)) \in \mathcal{V}_9$. Then there exists a neighborhood of x_* , denoted by \mathcal{V}_{10} , and a constant $\rho_1 > 0$ such that for any $x \in \mathcal{V}_{10}$, it holds that*

$$F(x_{++}) \leq F(x) - \rho_1 \|v(x)\|_F^2,$$

where $x_+ = R_x(d(x))$ and $x_{++} = R_{x_+}(d(x_+))$.

The global convergence result becomes: $\liminf_{k \rightarrow \infty} \|v(x_k)\|_F = 0$.

A new interpretation of RPN

Lemma

Suppose the previous Assumptions hold. Then there exists a neighborhood of x_* , denoted by \mathcal{V}_5 , such that

$$u(x) = \operatorname{argmin}_{u \in T_x \mathcal{M}, \hat{u} = \hat{v}(x)} G_x(u) = \frac{1}{2} \langle u, \mathfrak{B}_x u \rangle + \nabla f(x)^T u + \mu \|x + u\|_1 \quad (1)$$

holds for any $x \in \mathcal{V}_5$.

- First, find the ManPG search direction $v(x)$;
- Fixed the entries that corresponds to the zero of $x + v$;
- Solve (1) for $u(x)$;

A new interpretation of RPN

Lemma

Suppose the previous Assumptions hold. Then there exists a neighborhood of x_* , denoted by \mathcal{V}_5 , such that

$$u(x) = \underset{u \in T_x \mathcal{M}, \hat{u} = \hat{v}(x)}{\operatorname{argmin}} G_x(u) = \frac{1}{2} \langle u, \mathfrak{B}_x u \rangle + \nabla f(x)^T u + \mu \|x + u\|_1 \quad (1)$$

holds for any $x \in \mathcal{V}_5$.

- \mathcal{M}_{sub} : submanifold of the intersection of \mathcal{M} and the sparse constraints;
- $\mathfrak{B}_x^{(11)}$ is the Riemannian Hessian at x with respect to \mathcal{M}_{sub} ;
- $u(x)$ is the Riemannian Newton direction on \mathcal{M}_{sub} ;

A new interpretation of RPN

Lemma

Suppose the previous Assumptions hold. Then there exists a neighborhood of x_* , denoted by \mathcal{V}_5 , such that

$$u(x) = \operatorname{argmin}_{u \in T_x \mathcal{M}, \hat{u} = \hat{v}(x)} G_x(u) = \frac{1}{2} \langle u, \mathfrak{B}_x u \rangle + \nabla f(x)^T u + \mu \|x + u\|_1 \quad (1)$$

holds for any $x \in \mathcal{V}_5$.

- \mathcal{M}_{sub} : submanifold of the intersection of \mathcal{M} and the sparse constraints;
- $\mathfrak{B}_x^{(11)}$ is the Riemannian Hessian at x with respect to \mathcal{M}_{sub} ;
- $u(x)$ is the Riemannian Newton direction on \mathcal{M}_{sub} ;

No counterpart in the Euclidean space.

A Riemannian proximal Newton-CG method

Numerical experiments: sparse PCA

Sparse PCA problem

$$\min_{X \in \text{St}(p, n)} -\text{trace}(X^T A^T A X) + \mu \|X\|_1,$$

where $A \in \mathbb{R}^{m \times n}$ is a data matrix and

$\text{St}(p, n) = \{X \in \mathbb{R}^{n \times p} \mid X^T X = I_p\}$ is the compact Stiefel manifold.

A Riemannian proximal Newton-CG method

Numerical experiments: sparse PCA

Table: An average result of 20 random runs for random data. Multiple values of n , p , and μ are used. The subscript k indicates a scale of 10^k .

(n, p, μ)	Algo	iter	Fval	$\ v(x_k)\ _F$	time	sparsity
(400, 8, 0.8)	ManPG	3416.15	-2.16_1	3.66_{-9}	2.69	0.63
(400, 8, 0.8)	ManPG-Ada	1281.55	-2.16_1	1.06_{-10}	1.21	0.63
(400, 8, 0.8)	ManPQN	1260.40	-2.16_1	9.83_{-11}	0.72	0.63
(400, 8, 0.8)	RPN-CG	204.85	-2.16_1	1.16_{-11}	0.37	0.63
(800, 8, 0.8)	ManPG	4232.80	-5.92_1	1.84_{-7}	3.56	0.48
(800, 8, 0.8)	ManPG-Ada	1867.05	-5.92_1	2.57_{-10}	1.80	0.48
(800, 8, 0.8)	ManPQN	1883.80	-5.92_1	1.22_{-10}	1.43	0.48
(800, 8, 0.8)	RPN-CG	215.05	-5.92_1	1.07_{-11}	0.60	0.48

A Riemannian proximal Newton-CG method

Numerical experiments: sparse PCA

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- Proximal gradient on Stiefel manifold: ManPG, ManPG-Ada [CMSZ20];
- Proximal quasi-Newton on Stiefel manifold: ManPQN [WY23];
- The proposed method: RPN-CG;

A Riemannian proximal Newton-CG method

Numerical experiments: sparse PCA

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- Stop criterion: $\text{iter} \geq 5000$ or $\|v(x)\|_F \leq 10^{-10}$;
- The entries of A are drawn from the standard normal distribution;
- Runs that converges to the same minimizer are reported;
- Support estimation: $(x + v(x))_i$ nonzero and $|(x)_i| \geq \|v(x)\|_F$;

A Riemannian proximal Newton-CG method

Numerical experiments: sparse PCA

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(800, 8, 0.8)	RPN-CG	215.05	-5.92 ₁	1.07 ₋₁₁	0.60	0.48

RPN-CG always stops due to $\|v\|_F \leq 10^{-10}$
and is the most efficient one.

A Riemannian proximal Newton-CG method

Numerical experiments: sparse PCA

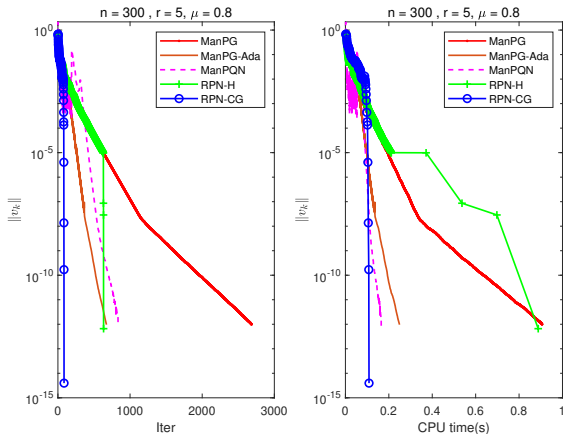


Figure: Sparse PCA: plots of $\|v(x_k)\|$ versus iterations and CPU times respectively.

- Riemannian optimization;
- Applications;
 - An example on an embedded submanifold;
 - An example on a quotient manifold;
- Smooth optimization framework;
 - Search direction/Riemannian metric;
 - Riemannian gradient/Hessian;
 - Retraction/vector transport;
- Research foci of Riemannian optimization;
 - Manifold recognition/structures;
 - Algorithm generalizations;
 - Applications/Libraries;
- A Riemannian proximal Newton-CG method;
 - A Riemannian proximal Newton method;
 - Truncated conjugate gradient;
 - Superlinear convergence approach;
 - Numerical experiments;
- Summary;

Thank you

Thank you!

References I



Ognjen Arandjelovic, Gregory Shakhnarovich, John Fisher, Roberto Cipolla, and Trevor Darrell.

Face recognition with image sets using manifold density divergence.

In *Computer Vision and Pattern Recognition, 2005. CVPR 2005. IEEE Computer Society Conference on*, volume 1, pages 581–588. IEEE, 2005.



Matthias Bollh ofer, Aryan Eftekhari, Simon Scheidegger, and Olaf Schenk.

Large-scale sparse inverse covariance matrix estimation.

SIAM Journal on Scientific Computing, 41(1):A380–A401, 2019.



Shixiang Chen, Shiqian Ma, Anthony Man-Cho So, and Tong Zhang.

Proximal gradient method for nonsmooth optimization over the Stiefel manifold.

SIAM Journal on Optimization, 30(1):210–239, 2020.



Haoran Chen, Yanfeng Sun, Junbin Gao, Yongli Hu, and Baocai Yin.

Fast optimization algorithm on riemannian manifolds and its application in low-rank learning.

Neurocomputing, 291:59 – 70, 2018.



Guang Cheng, Hesamoddin Salehian, and Baba Vemuri.

Efficient recursive algorithms for computing the mean diffusion tensor and applications to DTI segmentation.

Computer Vision—ECCV 2012, pages 390–401, 2012.



H. Drira, B. Ben Amor, A. Srivastava, M. Daoudi, and R. Slama.

3D face recognition under expressions, occlusions, and pose variations.

Pattern Analysis and Machine Intelligence, IEEE Transactions on, 35(9):2270–2283, 2013.



P. T. Fletcher and S. Joshi.

Riemannian geometry for the statistical analysis of diffusion tensor data.

Signal Processing, 87(2):250–262, 2007.



W. Huang, K. A. Gallivan, Anuj Srivastava, and P.-A. Absil.

Riemannian optimization for registration of curves in elastic shape analysis.

Journal of Mathematical Imaging and Vision, 54(3):320–343, 2015.

DOI:10.1007/s10851-015-0606-8.



Wen Huang, Meng Wei, Kyle A. Gallivan, and Paul Van Dooren.
A Riemannian Optimization Approach to Clustering Problems, 2022.



Zhiwu Huang, Ruiping Wang, Shiguang Shan, and Xilin Chen.
Face recognition on large-scale video in the wild with hybrid Euclidean-and-Riemannian metric learning.
Pattern Recognition, 48(10):3113–3124, 2015.



H. Laga, S. Kurtek, A. Srivastava, M. Golzarian, and S. J. Miklavcic.
A Riemannian elastic metric for shape-based plant leaf classification.
2012 International Conference on Digital Image Computing Techniques and Applications (DICTA), pages 1–7, December 2012.
doi:10.1109/DICTA.2012.6411702.



Jason D Lee, Yuekai Sun, and Michael A Saunders.
Proximal newton-type methods for minimizing composite functions.
SIAM Journal on Optimization, 24(3):1420–1443, 2014.



Jiwen Lu, Gang Wang, and Pierre Moulin.
Image set classification using holistic multiple order statistics features and localized multi-kernel metric learning.
In Proceedings of the IEEE International Conference on Computer Vision, pages 329–336, 2013.



Vidvuds Ozoliņš, Rongjie Lai, Russel Caffisch, and Stanley Osher.
Compressed modes for variational problems in mathematics and physics.
Proceedings of the National Academy of Sciences, 110(46):18368–18373, 2013.



Y. Rathi, A. Tannenbaum, and O. Michailovich.
Segmenting images on the tensor manifold.
In IEEE Conference on Computer Vision and Pattern Recognition, pages 1–8, June 2007.



Wutao Si, P. A. Absil, Wen Huang, Rujun Jiang, and Simon Vary.
A Riemannian Proximal Newton Method, 2024.
arXiv: 2304.04032v3.



Wutao Si, P.-A. Absil, Wen Huang, Rujun Jiang, and Simon Vary.

A Riemannian Proximal Newton Method.
SIAM Journal on Optimization, 34(1):654–681, 2024.



Gregory Shakhnarovich, John W Fisher, and Trevor Darrell.

Face recognition from long-term observations.
In *European Conference on Computer Vision*, pages 851–865. Springer, 2002.



Oncel Tuzel, Fatih Porikli, and Peter Meer.

Region covariance: A fast descriptor for detection and classification.
In *European conference on computer vision*, pages 589–600. Springer, 2006.



Qinsi Wang and Weihong Yang.

Proximal quasi-Newton method for composite optimization over the Stiefel manifold.
Journal of Scientific Computing, 95, 5 2023.



Hui Zou, Trevor Hastie, and Robert Tibshirani.

Sparse principal component analysis.
Journal of Computational and Graphical Statistics, 15(2):265–286, 2006.



Y. Zhang, Y. Lau, H.-W. Kuo, S. Cheung, A. Pasupathy, and J. Wright.

On the global geometry of sphere-constrained sparse blind deconvolution.
In *Proceedings of IEEE Conference on Computer Vision and Pattern Recognition (CVPR)*, 2017.