# Riemannian Optimization with its Application to Blind Deconvolution Problem

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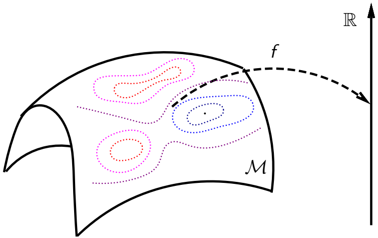
# Riemannian Optimization

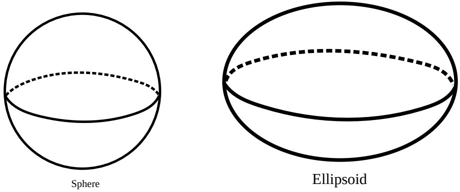
**Problem:** Given $f(x) : \mathcal{M} \to \mathbb{R}$, solve

$$\min_{x \in \mathcal{M}} f(x)$$

where $\mathcal{M}$ is a Riemannian manifold.



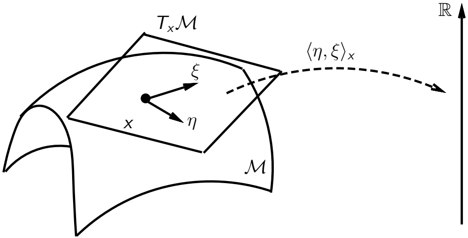
# Examples of Manifolds




- Stiefel manifold: $\mathrm{St}(p, n) = \{X \in \mathbb{R}^{n \times p} | X^T X = I_p\}$
- Grassmann manifold: Set of all $p$-dimensional subspaces of $\mathbb{R}^n$
- Set of fixed rank $m$-by-$n$ matrices
- And many more

# Riemannian Manifolds

Roughly, a Riemannian manifold $\mathcal{M}$ is a smooth set with a smoothly-varying inner product on the tangent spaces.

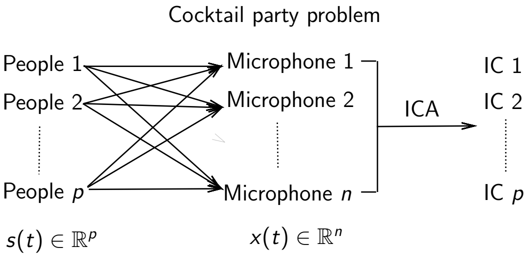


# Applications

Three applications are used to demonstrate the importance of the Riemannian optimization:

- Independent component analysis [CS93]
- Matrix completion problem [Van13, HAGH16]
- Elastic shape analysis of curves [SKJJ11, HGSA15]

# Application: Independent Component Analysis




- Observed signal is $x(t) = As(t)$
- One approach:
  - Assumption: $E\{s(t)s(t+\tau)\}$ is diagonal for all $\tau$
  - $C_\tau(x) := E\{x(t)x(x+\tau)^T\} = AE\{s(t)s(t+\tau)^T\}A^T$

# Application: Independent Component Analysis

- Minimize joint diagonalization cost function on the Stiefel manifold [TI06]:

$$f : \mathrm{St}(p, n) \to \mathbb{R} : V \mapsto \sum_{i=1}^{N} \|V^T C_i V - \mathrm{diag}(V^T C_i V)\|_F^2.$$

- $C_1, \ldots, C_N$ are covariance matrices and $\mathrm{St}(p, n) = \{X \in \mathbb{R}^{n \times p} | X^T X = I_p\}$.

# Application: Matrix Completion Problem

Matrix completion problem

$$
\begin{array}{c}
\begin{array}{cccccc}
\text{Movie } 1 & \text{Movie } 2 & & & & \text{Movie } n
\end{array}
\\
\begin{array}{c}
\text{User } 1 \\ \text{User } 2 \\ \\ \text{User } m
\end{array}
\begin{pmatrix}
 & 1 & & & 4 & \\
3 & 5 & & & 4 & \\
 & & 5 & 1 & & \\
 & 2 & & & 5 & 3
\end{pmatrix}
\end{array}
$$

Rate matrix $M$

- The matrix $M$ is sparse
- The goal: complete the matrix $M$

# Application: Matrix Completion Problem

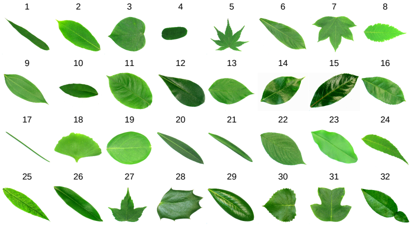
$$
\begin{array}{c}
\text{movies} \\
\begin{pmatrix}
a_{11} & & & a_{14} \\
 & & & a_{24} \\
 & & a_{33} & \\
a_{41} & & & \\
 & a_{52} & a_{53} &
\end{pmatrix}
\end{array}
=
\begin{array}{c}
\text{meta-user} \\
\begin{pmatrix}
b_{11} & b_{12} \\
b_{21} & b_{22} \\
b_{31} & b_{32} \\
b_{41} & b_{42} \\
b_{51} & b_{52}
\end{pmatrix}
\end{array}
\begin{array}{c}
\text{meta-movie} \\
\begin{pmatrix}
c_{11} & c_{12} & c_{13} & c_{14} \\
c_{21} & c_{22} & c_{23} & c_{24}
\end{pmatrix}
\end{array}
$$

- Minimize the cost function

$$f : \mathbb{R}_r^{m \times n} \to \mathbb{R} : X \mapsto f(X) = \|P_\Omega M - P_\Omega X\|_F^2.$$

- $\mathbb{R}_r^{m \times n}$ is the set of $m$-by-$n$ matrices with rank $r$. It is known to be a Riemannian manifold.

# Application: Elastic Shape Analysis of Curves




- Classification [LKS+12, HGSA15]
- Face recognition [DBS+13]

# Application: Elastic Shape Analysis of Curves

- Elastic shape analysis invariants:
    - Rescaling
    - Translation
    - Rotation
    - Reparametrization
- The shape space is a quotient space

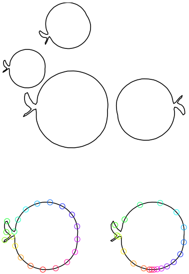
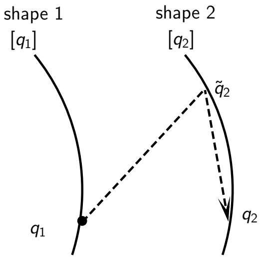


Figure: All are the same shape.

# Application: Elastic Shape Analysis of Curves



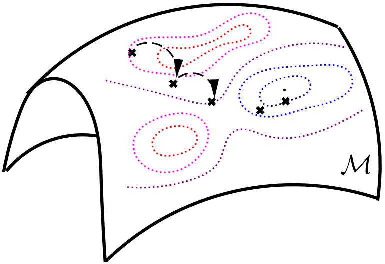

- Optimization problem $\min_{q_2 \in [q_2]} \mathrm{dist}(q_1, q_2)$ is defined on a Riemannian manifold
- Computation of a geodesic between two shapes
- Computation of Karcher mean of a population of shapes

# More Applications

- Role model extraction [MHB+16]
- Computations on SPD matrices [YHAG17]
- Phase retrieval problem [HGZ17]
- Blind deconvolution [HH17]
- Synchronization of rotations [Hua13]
- Computations on low-rank tensor
- Low-rank approximate solution for Lyapunov equation

# Comparison with Constrained Optimization

- All iterates on the manifold
- Convergence properties of unconstrained optimization algorithms
- No need to consider Lagrange multipliers or penalty functions
- Exploit the structure of the constrained set



# Iterations on the Manifold

Consider the following generic update for an iterative Euclidean optimization algorithm:

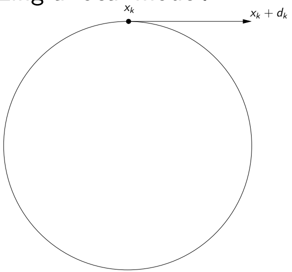
$$x_{k+1} = x_k + \Delta x_k = x_k + \alpha_k s_k \ .$$

This iteration is implemented in numerous ways, e.g.:

- Steepest descent: $x_{k+1} = x_k - \alpha_k \nabla f(x_k)$
- Newton's method: $x_{k+1} = x_k - \left[\nabla^2 f(x_k)\right]^{-1} \nabla f(x_k)$
- Trust region method: $\Delta x_k$ is set by optimizing a local model.

## Riemannian Manifolds Provide

- Riemannian concepts describing directions and movement on the manifold
- Riemannian analogues for gradient and Hessian



# Riemannian gradient and Riemannian Hessian

## Definition

The Riemannian gradient of $f$ at $x$ is the unique tangent vector in $T_x M$ satisfying $\forall \eta \in T_x M$, the directional derivative

$$\mathrm{D} f(x)[\eta] = \langle \operatorname{grad} f(x), \eta \rangle$$

and $\operatorname{grad} f(x)$ is the direction of steepest ascent.

## Definition

The Riemannian Hessian of $f$ at $x$ is a symmetric linear operator from $T_x M$ to $T_x M$ defined as

$$\operatorname{Hess} f(x) : T_x M \to T_x M : \eta \to \nabla_\eta \operatorname{grad} f,$$

where $\nabla$ is the affine connection.

# Retractions

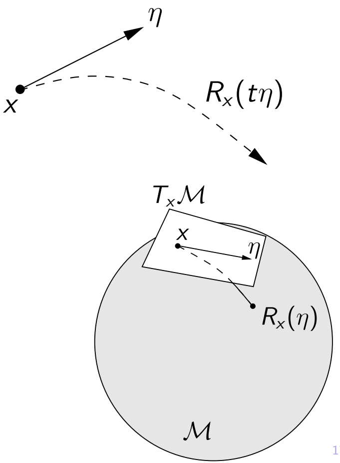
| Euclidean | Riemannian |
|---|---|
| $x_{k+1} = x_k + \alpha_k d_k$ | $x_{k+1} = R_{x_k}(\alpha_k \eta_k)$ |

## Definition

A retraction is a mapping $R$ from $TM$ to $M$ satisfying the following:

- $R$ is continuously differentiable
- $R_x(0) = x$
- $\mathrm{D}\, R_x(0)[\eta] = \eta$

- maps tangent vectors back to the manifold
- defines curves in a direction



# Categories of Riemannian optimization methods

**Retraction-based: local information only**

Line search-based: use local tangent vector and $R_x(t\eta)$ to define line

- Steepest decent
- Newton

Local model-based: series of flat space problems

- Riemannian trust region Newton (RTR)
- Riemannian adaptive cubic overestimation (RACO)

# Categories of Riemannian optimization methods

**Retraction and transport-based: information from multiple tangent spaces**

- Nonlinear conjugate gradient: multiple tangent vectors
- Quasi-Newton e.g. Riemannian BFGS: transport operators between tangent spaces

Additional element required for optimizing a cost function $(M, g)$:

- formulas for combining information from multiple tangent spaces.

# Vector Transports

## Vector Transport

- Vector transport: Transport a tangent vector from one tangent space to another
- $\mathcal{T}_{\eta_x}\xi_x$, denotes transport of $\xi_x$ to tangent space of $R_x(\eta_x)$. $R$ is a retraction associated with $\mathcal{T}$

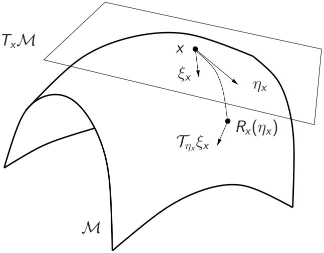



Figure: Vector transport.

# Retraction/Transport-based Riemannian Optimization

Given a retraction and a vector transport, we can generalize many Euclidean methods to the Riemannian setting. Do the Riemannian versions of the methods work well?

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Given a retraction and a vector transport, we can generalize many Euclidean methods to the Riemannian setting. Do the Riemannian versions of the methods work well?

No

- Lose many theoretical results and important properties;
- Impose restrictions on retraction/vector transport;

# Retraction/Transport-based Riemannian Optimization

## Benefits

- Increased generality does not compromise the important theory
- Less expensive than or similar to previous approaches
- May provide theory to explain behavior of algorithms specifically developed for a particular application – or closely related ones

## Possible Problems

- May be inefficient compared to algorithms that exploit application details

# Some History of Optimization On Manifolds (I)

Luenberger (1973), *Introduction to linear and nonlinear programming*. Luenberger mentions the idea of performing line search along geodesics, "which we would use if it were computationally feasible (which it definitely is not)". Rosen (1961) essentially anticipated this but was not explicit in his Gradient Projection Algorithm.

Gabay (1982), *Minimizing a differentiable function over a differential manifold*. Steepest descent along geodesics; Newton's method along geodesics; Quasi-Newton methods along geodesics. On Riemannian submanifolds of $\mathbb{R}^n$.

Smith (1993-94), *Optimization techniques on Riemannian manifolds*. Levi-Civita connection $\nabla$; Riemannian exponential mapping; parallel translation.

# Some History of Optimization On Manifolds (II)

The "pragmatic era" begins:

Manton (2002), *Optimization algorithms exploiting unitary constraints* "The present paper breaks with tradition by not moving along geodesics". The geodesic update $\mathrm{Exp}_x \eta$ is replaced by a projective update $\pi(x+\eta)$, the *projection* of the point $x+\eta$ onto the manifold.

Adler, Dedieu, Shub, et al. (2002), *Newton's method on Riemannian manifolds and a geometric model for the human spine*. The exponential update is relaxed to the general notion of *retraction*. The geodesic can be replaced by any (smoothly prescribed) curve tangent to the search direction.

Absil, Mahony, Sepulchre (2007) Nonlinear conjugate gradient using retractions.

# Some History of Optimization On Manifolds (III)

Theory, efficiency, and library design improve dramatically:

Absil, Baker, Gallivan (2004-07), Theory and implementations of Riemannian Trust Region method. Retraction-based approach. Matrix manifold problems, software repository

`http://www.math.fsu.edu/~cbaker/GenRTR`

Anasazi Eigenproblem package in Trilinos Library at Sandia National Laboratory

Absil, Gallivan, Qi (2007-10), Basic theory and implementations of Riemannian BFGS and Riemannian Adaptive Cubic Overestimation. Parallel translation and Exponential map theory, Retraction and vector transport empirical evidence.

# Some History of Optimization On Manifolds (IV)

Ring and With (2012), combination of differentiated retraction and isometric vector transport for convergence analysis of RBFGS

Absil, Gallivan, Huang (2009-2017), Complete theory of Riemannian Quasi-Newton and related transport/retraction conditions, Riemannian SR1 with trust-region, RBFGS on partly smooth problems, A C++ library: `http://www.math.fsu.edu/~whuang2/ROPTLIB`

Sato, Iwai (2013-2015), Zhu (2017), Global convergence analysis for Riemannian conjugate gradient methods

Bonnabel (2011), Sato, Kasai, Mishra(2017) Riemannian stochastic gradient descent method.

Many people Application interests increase noticeably

# Current UCL/FSU Methods

- Riemannian Steepest Descent [AMS08]
- Riemannian conjugate gradient [AMS08]
- Riemannian Trust Region Newton [ABG07]: global, quadratic convergence
- Riemannian Broyden Family [HGA15, HAG18] : global (convex), superlinear convergence
- Riemannian Trust Region SR1 [HAG15]: global, $(d+1)-$superlinear convergence
- For large problems
  - Limited memory RTRSR1
  - Limited memory RBFGS

# Current UCL/FSU Methods

Riemannian manifold optimization library (ROPTLIB) is used to optimize a function on a manifold.

- Most state-of-the-art methods;
- Commonly-encountered manifolds;
- Written in C++;
- Interfaces with Matlab, Julia and R;
- BLAS and LAPACK;
- `www.math.fsu.edu/~whuang2/Indices/index_ROPTLIB.html`

# Current/Future Work on Riemannian methods

- Manifold and inequality constraints
- Discretization of infinite dimensional manifolds and the convergence/accuracy of the approximate minimizers – specific to a problem and extracting general conclusions
- Partly smooth cost functions on Riemannian manifold
- Limited-memory quasi-Newton methods on manifolds

# Blind deconvolution

## [Blind deconvolution]

Blind deconvolution is to recover two unknown signals from their convolution.

Blurred image

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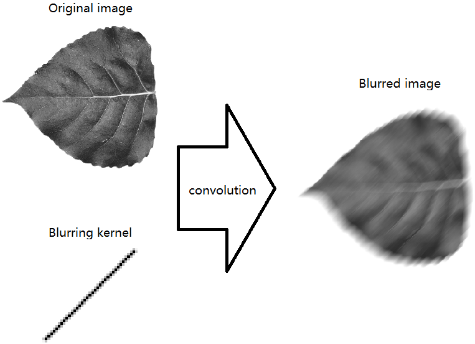

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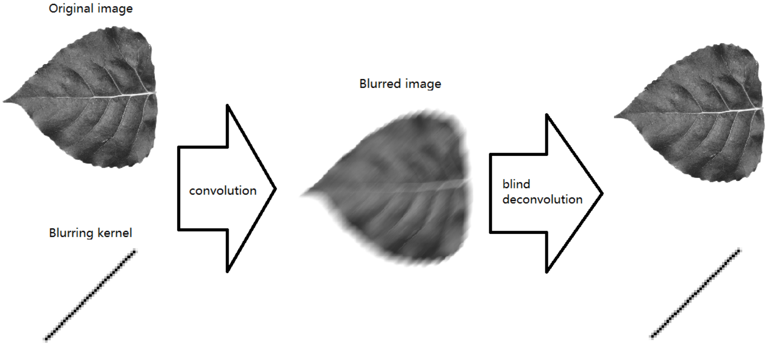

# Problem Statement

## [Blind deconvolution (Discretized version)]

Blind deconvolution is to recover two unknown signals $\mathbf{w} \in \mathbb{C}^L$ and $\mathbf{x} \in \mathbb{C}^L$ from their convolution $\mathbf{y} = \mathbf{w} * \mathbf{x} \in \mathbb{C}^L$.

- We only consider circular convolution:

$$\begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \mathbf{y}_3 \\ \vdots \\ \mathbf{y}_L \end{bmatrix} = \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_L & \mathbf{w}_{L-1} & \dots & \mathbf{w}_2 \\ \mathbf{w}_2 & \mathbf{w}_1 & \mathbf{w}_L & \dots & \mathbf{w}_3 \\ \mathbf{w}_3 & \mathbf{w}_2 & \mathbf{w}_1 & \dots & \mathbf{w}_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{w}_L & \mathbf{w}_{L-1} & \mathbf{w}_{L-2} & \dots & \mathbf{w}_1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \\ \vdots \\ \mathbf{x}_L \end{bmatrix}$$

- Let $y = \mathbf{F}\mathbf{y}$, $w = \mathbf{F}\mathbf{w}$, and $x = \mathbf{F}\mathbf{x}$, where $\mathbf{F}$ is the DFT matrix;
- $y = w \odot x$, where $\odot$ is the Hadamard product, i.e., $y_i = w_i x_i$.
- Equivalent question: Given $y$, find $w$ and $x$.

# Problem Statement

**Problem**: Given $y \in \mathbb{C}^L$, find $w, x \in \mathbb{C}^L$ so that $y = w \odot x$.

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- Assumption: $w$ and $x$ are in known subspaces, i.e., $w = Bh$ and $x = \overline{Cm}$, $B \in \mathbb{C}^{L \times K}$ and $C \in \mathbb{C}^{L \times N}$;

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  - Reasonable in various applications;
  - Leads to mathematical rigor; $(L/(K+N)$ reasonably large)

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## Problem under the assumption

Given $y \in \mathbb{C}^L$, $B \in \mathbb{C}^{L \times K}$ and $C \in \mathbb{C}^{L \times N}$, find $h \in \mathbb{C}^K$ and $m \in \mathbb{C}^N$ so that

$$y = Bh \odot \overline{Cm} = \mathrm{diag}(Bhm^* C^*).$$

# Related work

$$\boxed{\text{Find } h, m, \text{ s. t. } y = \mathrm{diag}(Bhm^* C^*);}$$

- Ahmed et al. [ARR14][1]
  - Convex problem:

$$\min_{X \in \mathbb{C}^{K \times N}} \|X\|_n, \text{ s. t. } y = \mathrm{diag}(BXC^*),$$

where $\|\cdot\|_n$ denotes the nuclear norm, and $X = hm^*$;

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    where $\|\cdot\|_n$ denotes the nuclear norm, and $X = hm^*$;
  - (Theoretical result): the unique minimizer $\overset{\text{high probability}}{=\!=\!=}$ the true solution;
  - The convex problem is expensive to solve;

[1] A. Ahmed, B. Recht, and J. Romberg, Blind deconvolution using convex programming, *IEEE Transactions on Information Theory*, 60:1711-1732, 2014

# Related work

$$\boxed{\text{Find } h, m, \text{ s. t. } y = \mathrm{diag}(Bhm^* C^*);}$$

- Li et al. [LLSW16][2]
  - Nonconvex problem[3]:

$$\min_{(h,m) \in \mathbb{C}^K \times \mathbb{C}^N} \|y - \mathrm{diag}(Bhm^* C^*)\|_2^2;$$

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[2] X. Li et. al., Rapid, robust, and reliable blind deconvolution via nonconvex optimization, *preprint arXiv:1606.04933*, 2016

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  - (Theoretical result):
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  - (Theoretical result):
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  - Lower successful recovery probability than alternating minimization algorithm empirically.

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# Manifold Approach

$$\boxed{\text{Find } h, m, \text{ s. t. } y = \mathrm{diag}(Bhm^*C^*);}$$

- The problem is defined on the set of rank-one matrices (denoted by $\mathbb{C}_1^{K\times N}$), neither $\mathbb{C}^{K\times N}$ nor $\mathbb{C}^K \times \mathbb{C}^N$; Why not work on the manifold directly?

# Manifold Approach

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- A representative Riemannian method: Riemannian steepest descent method (RSD)
  - A good initialization
  - (RSD + the good initialization) $\xRightarrow{\text{high probability}}$ the true solution;
  - The Riemannian Hessian at the true solution is well-conditioned;

# Manifold Approach

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- The problem is defined on the set of rank-one matrices (denoted by $\mathbb{C}_1^{K \times N}$), neither $\mathbb{C}^{K \times N}$ nor $\mathbb{C}^K \times \mathbb{C}^N$; Why not work on the manifold directly?
- Optimization on manifolds: A Riemannian steepest descent method;
  - Representation of $\mathbb{C}_1^{K \times N}$;
  - Representation of directions (tangent vectors);
  - Riemannian metric;
  - Riemannian gradient;

# A Representation of $\mathbb{C}_1^{K\times N}$: $\mathbb{C}_*^K \times \mathbb{C}_*^N / \mathbb{C}_*$

- Given $X \in \mathbb{C}_1^{K\times N}$, there exists $(h, m)$, $h \neq 0$ and $m \neq 0$ such that $X = hm^*$;
- $(h, m)$ is not unique;
- The equivalent class: $[(h, m)] = \{(ha, ma^{-*}) \mid a \neq 0\}$;
- Quotient manifold: $\mathbb{C}_*^K \times \mathbb{C}_*^N / \mathbb{C}_* = \{[(h, m)] \mid (h, m) \in \mathbb{C}_*^K \times \mathbb{C}_*^N\}$

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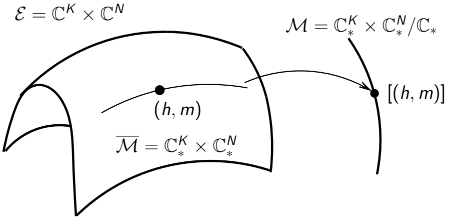



$$\mathbb{C}_*^K \times \mathbb{C}_*^N/\mathbb{C}_* \simeq \mathbb{C}_1^{K\times N}$$

# A Representation of $\mathbb{C}_1^{K\times N}$: $\mathbb{C}_*^K \times \mathbb{C}_*^N/\mathbb{C}_*$

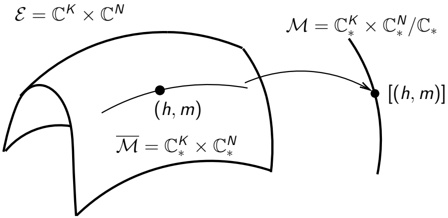
Cost function[4]

- Riemannian approach:

  $$f : \mathbb{C}_*^K \times \mathbb{C}_*^N/\mathbb{C}_* \to \mathbb{R} : [(h,m)] \mapsto \|y - \mathrm{diag}(Bhm^* C^*)\|_2^2.$$

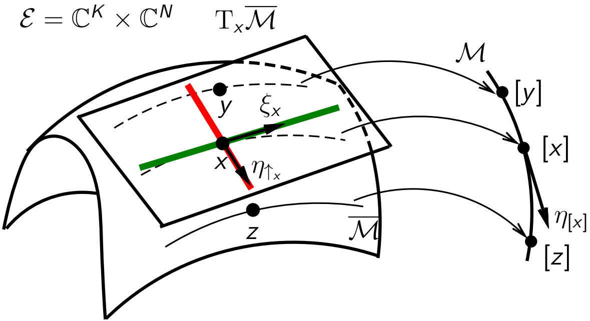
- Approach in [LLSW16]:

  $$\mathfrak{f} : \mathbb{C}^K \times \mathbb{C}^N \to \mathbb{R} : (h,m) \mapsto \|y - \mathrm{diag}(Bhm^* C^*)\|_2^2.$$




[4] The penalty in the cost function is not added for simplicity.

# Representation of directions on $\mathbb{C}_*^K \times \mathbb{C}_*^N / \mathbb{C}_*$



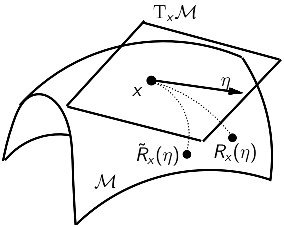

- $x$ denotes $(h, m)$;
- Green line: the tangent space of $[x]$;
- Red line (horizontal space at $x$): orthogonal to the green line;
- Horizontal space at $x$: a representation of the tangent space of $\mathcal{M}$ at $[x]$;

# Retraction

| Euclidean | Riemannian |
|---|---|
| $x_{k+1} = x_k + \alpha_k d_k$ | $x_{k+1} = R_{x_k}(\alpha_k \eta_k)$ |

- Retraction: $R : \mathrm{T}\,\mathcal{M} \to \mathcal{M}$
- $R(0_{[x]}) = [x]$
- $\frac{dR(t\eta_{[x]})}{dt}\big|_{t=0} = \eta_{[x]}$;
- Retraction on $\mathbb{C}_*^K \times \mathbb{C}_*^N / \mathbb{C}_*$:

$$R_{[(h,m)]}(\eta_{[(h,m)]}) = [(h + \eta_h, m + \eta_m)]\,.$$




Two retractions: $R$ and $\tilde{R}$

# A Riemannian metric

Riemannian metric:

- Inner product on tangent spaces
- Define angles and lengths

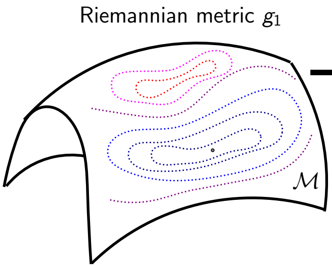



Figure: Changing metric may influence the difficulty of a problem.

# A Riemannian metric

$$\boxed{\min_{[(h,m)]} \|y - \mathrm{diag}(Bhm^* C^*)\|_2^2}$$

## Idea for choosing a Riemannian metric

The block diagonal terms in the Euclidean Hessian are used to choose the Riemannian metric.

# A Riemannian metric

$$\boxed{\min_{[(h,m)]} \|y - \mathrm{diag}(Bhm^* C^*)\|_2^2}$$

**Idea for choosing a Riemannian metric**

The block diagonal terms in the Euclidean Hessian are used to choose the Riemannian metric.

- Let $\langle u, v \rangle_2 = \mathrm{Re}(\mathrm{trace}(u^* v))$:

$$\frac{1}{2}\langle \eta_h, \mathrm{Hess}_h f[\xi_h] \rangle_2 = \langle \mathrm{diag}(B\eta_h m^* C^*), \mathrm{diag}(B\xi_h m^* C^*) \rangle_2 \approx \langle \eta_h m^*, \xi_h m^* \rangle_2$$
$$\frac{1}{2}\langle \eta_m, \mathrm{Hess}_m f[\xi_m] \rangle_2 = \langle \mathrm{diag}(Bh\eta_m^* C^*), \mathrm{diag}(Bh\xi_m^* C^*) \rangle_2 \approx \langle h\eta_m^*, h\xi_m^* \rangle_2,$$

where $\approx$ can be derived from some assumptions;

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  where $\approx$ can be derived from some assumptions;
- The Riemannian metric:

$$g\left(\eta_{[x]}, \xi_{[x]}\right) = \langle \eta_h, \xi_h m^* m \rangle_2 + \langle \eta_m^*, \xi_m^* h^* h \rangle_2;$$

# Riemannian gradient

- Riemannian gradient
  - A tangent vector: $\mathrm{grad}\, \mathbf{f}([x]) \in \mathrm{T}_{[x]}\, \mathcal{M}$;
  - Satisfies: $\mathrm{D} f([x])[\eta_{[x]}] = g(\mathrm{grad}\, f([x]), \eta_{[x]}), \;\; \forall \eta_{[x]} \in \mathrm{T}_{[x]}\, \mathcal{M}$;
- Represented by a vector in a horizontal space;
- Riemannian gradient:

$$(\mathrm{grad}\, f([(h,m)]))_{\uparrow_{(h,m)}} = Proj\left(\nabla_h f(h,m)(m^* m)^{-1}, \nabla_m f(h,m)(h^* h)^{-1}\right);$$

# A Riemannian steepest descent method (RSD)

An implementation of a Riemannian steepest descent method[5]

0. Given $(h_0, m_0)$, step size $\alpha > 0$, and set $k = 0$
1. $$d_k = \|h_k\|_2 \|m_k\|_2, \; h_k \leftarrow \sqrt{d_k} \frac{h_k}{\|h_k\|_2}; \; m_k \leftarrow \sqrt{d_k} \frac{m_k}{\|m_k\|_2};$$
2. $$(h_{k+1}, m_{k+1}) = (h_k, m_k) - \alpha \left( \frac{\nabla_{h_k} f(h_k, m_k)}{d_k}, \frac{\nabla_{m_k} f(h_k, m_k)}{d_k} \right);$$
3. If not converge, goto Step 2.

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Wirtinger flow Method in [LLSW16]

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# Penalty

Penalty term for (i) Riemannian method, (ii) Wirtinger flow [LLSW16]

$$\text{(i):}\ \rho \sum_{i=1}^{L} G_0\left(\frac{L|b_i^* h|^2 \|m\|_2^2}{8d^2\mu^2}\right)$$

$$\text{(ii):}\ \rho \left[ G_0\left(\frac{\|h\|_2^2}{2d}\right) + G_0\left(\frac{\|m\|_2^2}{2d}\right) + \sum_{i=1}^{L} G_0\left(\frac{L|b_i^* h|^2}{8d\mu^2}\right)\right],$$

where $G_0(t) = \max(t-1, 0)^2$, $[b_1 b_2 \dots b_L]^* = B$.

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where $G_0(t) = \max(t-1, 0)^2$, $[b_1 b_2 \dots b_L]^* = B$.

- The first two terms in (ii) penalize large values of $\|h\|_2$ and $\|m\|_2$;
- The other terms promote a small coherence;
- The one in (i) is defined in the quotient space whereas the one in (ii) is not.

# Penalty/Coherence

- Coherence is defined as

$$\mu_h^2 = \frac{L\|Bh\|_\infty^2}{\|h\|_2^2} = \frac{L \max\left(|b_1^* h|^2, |b_2^* h|^2, \ldots, |b_L^* h|^2\right)}{\|h\|_2^2};$$

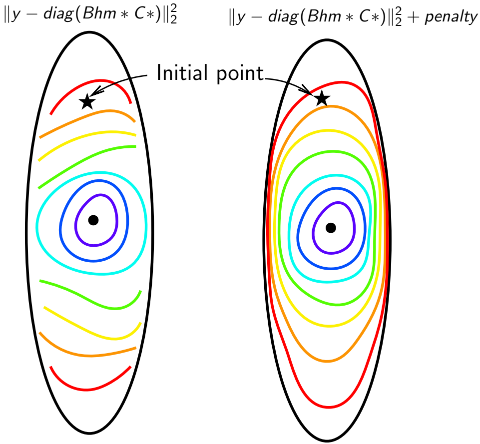
- Coherence at the true solution $[(h_\sharp, m_\sharp)]$
  - influences the probability of recovery
  - Small coherence is preferred

# Penalty

Promote low coherence:

$$\rho \sum_{i=1}^{L} G_0 \left( \frac{L |b_i^* h|^2 \|m\|_2^2}{8 d^2 \mu^2} \right),$$

where $G_0(t) = \max(t-1, 0)^2$;



# Initialization

Initialization method [LLSW16]

- $(d, \tilde{h}_0, \tilde{m}_0)$: SVD of $B^* \operatorname{diag}(y) C$;
- $h_0 = \operatorname{argmin}_z \|z - \sqrt{d}\tilde{h}_0\|_2^2$, subject to $\sqrt{L}\|Bz\|_\infty \leq 2\sqrt{d}\mu$;
- $m_0 = \sqrt{d}\tilde{m}_0$;
- Initial iterate $[(h_0, m_0)]$;

# Numerical Results

- Synthetic tests
  - Efficiency
  - Probability of successful recovery
- Image deblurring
  - Kernels with known supports
  - Motion kernel with unknown supports

# Synthetic tests

- The matrix $B$ is the first $K$ column of the unitary DFT matrix;
- The matrix $C$ is a Gaussian random matrix;
- The measurement $y = \mathrm{diag}(B h_\sharp m_\sharp^* C^*)$, where entries in the true solution $(h_\sharp, m_\sharp)$ are drawn from Gaussian distribution;
- All tested algorithms use the same initial point;
- Stop when $\|y - \mathrm{diag}(Bhm^* C^*)\|_2 / \|y\|_2 \leq 10^{-8}$;

# Efficiency

$$\boxed{\min \|y - \mathrm{diag}(Bhm^* C^*)\|_2^2}$$

Table: Comparisons of efficiency

| | $L = 400, K = N = 50$ | | | $L = 600, K = N = 50$ | | |
|---|---|---|---|---|---|---|
| Algorithms | [LLSW16] | [LWB13] | R-SD | [LLSW16] | [LWB13] | R-SD |
| $nBh/nCm$ | 351 | 718 | 208 | 162 | 294 | 122 |
| $nFFT$ | 870 | 1436 | 518 | 401 | 588 | 303 |
| $RMSE$ | $2.22_{-8}$ | $3.67_{-8}$ | $2.20_{-8}$ | $1.48_{-8}$ | $2.34_{-8}$ | $1.42_{-8}$ |

- An average of 100 random runs
- nBh/nCm: the numbers of $Bh$ and $Cm$ multiplication operations respectively
- nFFT: the number of Fourier transform
- RMSE: the relative error $\frac{\|hm^* - h_\sharp m_\sharp^*\|_F}{\|h_\sharp\|_2 \|m_\sharp\|_2}$

[LLSW16]: X. Li et. al., Rapid, robust, and reliable blind deconvolution via nonconvex optimization, *preprint arXiv:1606.04933*, 2016
[LWB13]: K. Lee et. al., Near Optimal Compressed Sensing of a Class of Sparse Low-Rank Matrices via Sparse Power Factorization *preprint arXiv:1312.0525*, 2013

# Probability of successful recovery

- Success if $\frac{\|hm^* - h_\sharp m_\sharp^*\|_F}{\|h_\sharp\|_2 \|m_\sharp\|_2} \leq 10^{-2}$

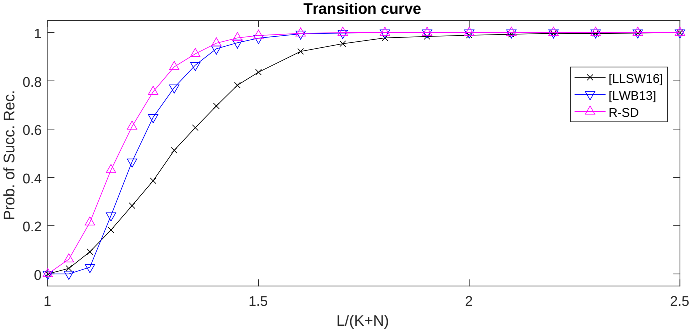



Figure: Empirical phase transition curves for 1000 random runs.

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[LLSW16]: X. Li et. al., Rapid, robust, and reliable blind deconvolution via nonconvex optimization, *preprint arXiv:1606.04933*, 2016

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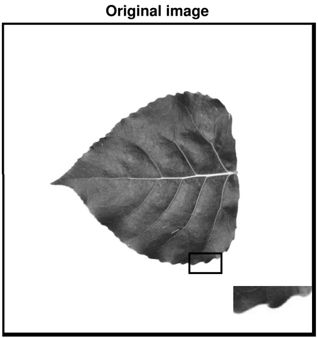
# Image deblurring




- Image [WBX$^+$07]: 1024-by-1024 pixels

# Image deblurring with various kernels

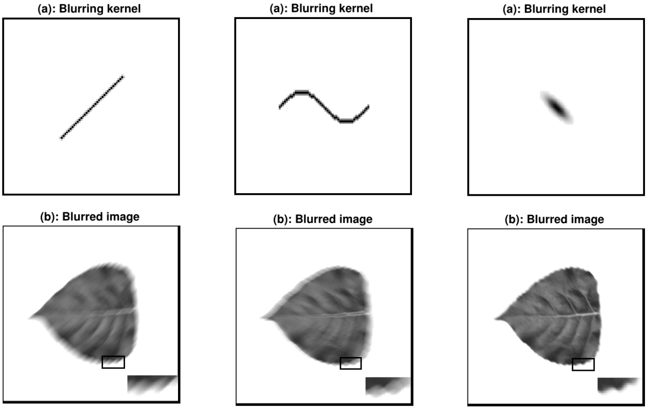


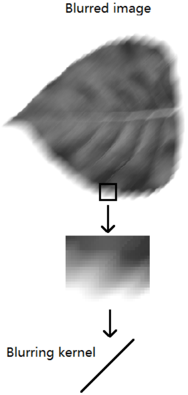

Figure: Left: Motion kernel by Matlab function "fspecial('motion', 50, 45)"; Middle: Kernel like function "sin"; Right: Gaussian kernel with covariance $[1, 0.8; 0.8, 1]$;

# Image deblurring with various kernels

$$\min \|y - \operatorname{diag}(Bhm^* C^*)\|_2^2$$

What subspaces are the two unknown signals in?

- Image is approximately sparse in the Haar wavelet basis
- Support of the blurring kernel is learned from the blurred image



# Image deblurring with various kernels

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What subspaces are the two unknown signals in?

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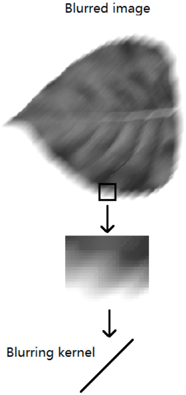
  Use the blurred image to learn the dominated basis vectors: **C**.

- Support of the blurring kernel is learned from the blurred image



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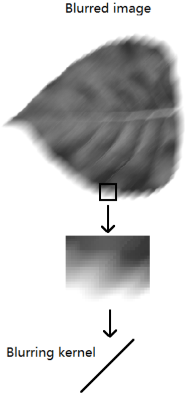
  Suppose the supports of the blurring kernels are known: **B**.



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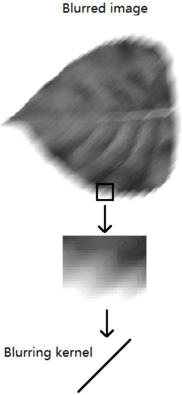
  Suppose the supports of the blurring kernels are known: **B**.

- $L = 1048576$, $N = 20000$, $K_{motion} = 109$, $K_{sin} = 153$, $K_{Gaussian} = 181$;



# Image deblurring with various kernels

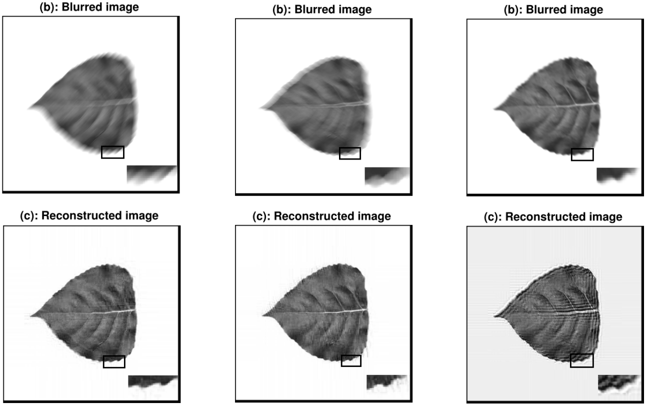



Figure: The number of iterations is 80; Computational times are about 48s; Relative errors $\left\|\hat{\mathbf{y}} - \frac{\|\mathbf{y}\|}{\|\mathbf{y}_f\|}\mathbf{y}_f\right\| / \|\hat{\mathbf{y}}\|$ are 0.038, 0.040, and 0.089 from left to right.

# Image deblurring with unknown supports

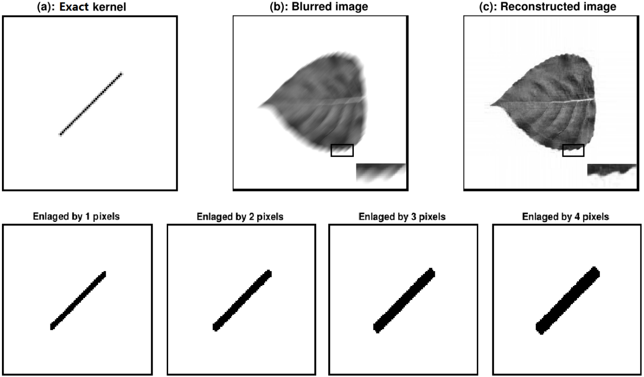



Figure: Top: reconstructed image using the exact support; Bottom: estimated supports with the numbers of nonzero entries: $K_1 = 183$, $K_2 = 265$, $K_3 = 351$, and $K_4 = 441$;

# Image deblurring with unknown supports

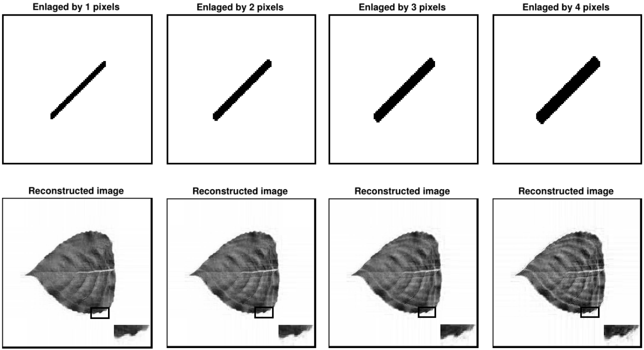



Figure: Relative errors $\left\|\hat{\mathbf{y}} - \frac{\|\mathbf{y}\|}{\|\mathbf{y}_f\|}\mathbf{y}_f\right\| / \|\hat{\mathbf{y}}\|$ are 0.044, 0.048, 0.052, and 0.067 from left to right.

# Summary

- Introduced the framework of Riemannian optimization
- Used applications to show the importance of Riemannian optimization
- Briefly reviewed the history of Riemannian optimization
- Introduced the blind deconvolution problem
- Reviewed related work
- Introduced a Riemannian steepest descent method
- Demonstrated the performance of the Riemannian steepest descent method

# Thank you

Thank you!

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