

# A Riemannian Accelerated Proximal Gradient Method

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Fuzhou

- 1 Riemannian Proximal Gradient Methods Review
- 2 A Riemannian Accelerated Proximal Gradient Method
- 3 Adaptive Restart for Riemannian Accelerated Proximal Gradient Method
- 4 Summary

# Outline

- 1 Riemannian Proximal Gradient Methods Review
  - Problem Statement
  - Related Work
- 2 A Riemannian Accelerated Proximal Gradient Method
- 3 Adaptive Restart for Riemannian Accelerated Proximal Gradient Method
- 4 Summary

# Problem Statement

Optimization on Manifolds with Structure:

$$\min_{x \in \mathcal{M}} F(x) = f(x) + h(x),$$

- $\mathcal{M}$  is a finite-dimensional Riemannian manifold;
- $f$  is smooth and may be nonconvex;
- $h$  is continuous but may be nonsmooth;

Applications:

- sparse principal component analysis [ZBDZ23, GHT15, ZHT06];
- clustering [HWGVD25, LYL16, PZ18];
- image processing [OSZ17, YG17];
- compressed modes [OLCO13];
- face expression recognition [FRJA18];

# Goal

Optimization on Manifolds with Structure:

$$\min_{x \in \mathcal{M}} F(x) = f(x) + h(x),$$

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Goal:

- Propose an accelerated proximal gradient algorithm on manifold;
- Provide a convergence analysis;

# Related Work

## Euclidean Setting

A proximal gradient method, initial iterate  $x_0$ :

$$\begin{cases} d_k = \arg \min_p \langle \nabla f(x_k), p \rangle + \frac{L}{2} \|p\|_{\mathbb{F}}^2 + h(x_k + p) & \text{(Proximal mapping)} \\ x_{k+1} = x_k + d_k & \text{(Update iterates)} \end{cases}$$

---

1. The update rule:  $x_{k+1} = \arg \min_x f(x_k) + \langle \nabla f(x_k), x - x_k \rangle + \frac{L}{2} \|x - x_k\|^2 + h(x)$ .

# Related Work

## Euclidean Setting

A proximal gradient method, initial iterate  $x_0$ :<sup>1</sup>

$$\begin{cases} d_k = \arg \min_p \langle \nabla f(x_k), p \rangle + \frac{L}{2} \|p\|_F^2 + h(x_k + p) & \text{(Proximal mapping)} \\ x_{k+1} = x_k + d_k & \text{(Update iterates)} \end{cases}$$

FISTA in convex [BT09]:

Given  $x_0$ , let  $y_0 = x_0, t_0 = 1$ ;

$$\begin{cases} d_{y_k} = \operatorname{argmin}_p \langle \nabla f(y_k), p \rangle + \frac{L}{2} \|p\|_F^2 + h(y_k + p) \\ x_{k+1} = y_k + d_{y_k} \\ t_{k+1} = \frac{\sqrt{4t_k^2 + 1} + 1}{2} \\ y_{k+1} = x_{k+1} + \frac{t_k - 1}{t_{k+1}} (x_{k+1} - x_k). \end{cases}$$

- Based on the Nesterov momentum technique;
- Two-point iterative sequence:  $x_k$  and  $y_k$ ;
- $O\left(\frac{1}{k^2}\right)$  sublinear convergence rate for convex  $f$  and  $h$ ;

1. The update rule:  $x_{k+1} = \arg \min_x f(x_k) + \langle \nabla f(x_k), x - x_k \rangle + \frac{L}{2} \|x - x_k\|^2 + h(x)$ .

# Related Work

## Euclidean Setting

FISTA in strongly convex [dST<sup>+</sup>21]:

Given  $x_0$ , let  $z_0 = x_0, A_0 = 0, q = \frac{\mu}{L}$  ( $\mu \geq 0$ );

$$\left\{ \begin{array}{l} A_{k+1} = \frac{2A_k + 1 + \sqrt{4A_k + 4qA_k^2 + 1}}{2(1-q)} \\ \tau_k = \frac{(A_{k+1} - A_k)(1 + qA_k)}{A_{k+1} + 2qA_kA_{k+1} - qA_k^2}, \quad \gamma_k = \frac{A_{k+1} - A_k}{1 + qA_{k+1}} \\ y_k = x_k + \tau_k(z_k - x_k) \\ d_{y_k} = \operatorname{argmin}_p \langle \nabla f(y_k), p \rangle + \frac{L}{2} \|p\|_F^2 + h(y_k + p) \\ x_{k+1} = y_k + d_{y_k} \\ z_{k+1} = (1 - q\gamma_k)z_k + q\gamma_k y_k + \gamma_k d_{y_k}. \end{array} \right.$$

- Three-point iterative sequence:  $x_k, y_k$  and  $z_k$ ;
- $\min\{O\left(\frac{1}{k^2}\right), O(1 - \sqrt{q})^k\}$  convergence rate for strongly convex  $f$  and convex  $h$ ;

[dST<sup>+</sup>21] A. d'Aspremont, D. Scieur and A. Taylor. Acceleration methods. Foundations and Trends in Optimization, 5(1-2): 1–245, 2021.

# Related Work

## Euclidean Setting

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Given  $x_0$ , let  $z_0 = x_0, A_0 = 0, q = \frac{\mu}{L}$  ( $\mu \geq 0$ );

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- Three-point iterative sequence:  $x_k, y_k$  and  $z_k$ ;
- $\min\{O\left(\frac{1}{k^2}\right), O(1 - \sqrt{q})^k\}$  convergence rate for strongly convex  $f$  and convex  $h$ ;
- A unified accelerated method;

[dST<sup>+</sup>21] A. d'Aspremont, D. Scieur and A. Taylor. Acceleration methods. Foundations and Trends in Optimization, 5(1-2): 1–245, 2021.

# Related Work

## Riemannian Setting

Optimization with Structure:

$$\min_{x \in \mathcal{M}} F(x) = f(x) + h(x),$$

---

- Proximal Gradient 1
- Proximal Gradient 2
- Proximal Gradient 3

How can these methods be extended to Riemannian manifolds?

# Related Work

## Riemannian Setting

Optimization with Structure:

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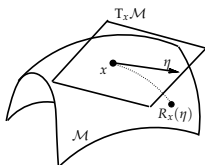
- Proximal Gradient 1

- Proximal Gradient 2

- Proximal Gradient 3

[CMSZ20]: Given  $x_0$ ,

$$\begin{cases} \eta_k = \arg \min_{\eta \in T_{x_k} \mathcal{M}} \langle \nabla f(x_k), \eta \rangle + \frac{L}{2} \|\eta\|_F^2 + h(x_k + \eta) \\ x_{k+1} = R_{x_k}(\alpha_k \eta_k) \text{ with an appropriate step size } \alpha_k; \end{cases}$$



[CMSZ20] S. Chen, S. Ma, A. Man-Cho So, and T. Zhang. Proximal gradient method for nonsmooth optimization over the Stiefel manifold. *SIAM Journal on Optimization*, 30(1):210-239, 2020.

# Related Work

## Riemannian Setting

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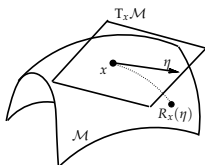
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- Direction in the tangent space;

- For embedded submanifold;

- Solved by a semismooth Newton method;

- Any limit point is a critical point;

- No local convergence rate results;

[CMSZ20] S. Chen, S. Ma, A. Man-Cho So, and T. Zhang. Proximal gradient method for nonsmooth optimization over the Stiefel manifold. *SIAM Journal on Optimization*, 30(1):210-239, 2020.

# Related Work

## Riemannian Setting

Optimization with Structure:

$$\min_{x \in \mathcal{M}} F(x) = f(x) + h(x),$$

- 
- [HW22b]: Given  $x_0$ ,
- Proximal Gradient 1
  - Proximal Gradient 2
  - Proximal Gradient 3
- $$\left\{ \begin{array}{l} \text{Let } \ell_{x_k}(\eta) = \langle \text{grad} f(x_k), \eta \rangle_{x_k} + \frac{L}{2} \|\eta\|_{x_k}^2 + h(R_{x_k}(\eta)); \\ \eta_k \text{ is a stationary point of } \ell_{x_k} \text{ on } \mathbb{T}_{x_k} \mathcal{M} \text{ and } \ell_{x_k}(0) \geq \ell_{x_k}(\eta_k); \\ x_{k+1} = R_{x_k}(\eta_k); \end{array} \right.$$

# Related Work

## Riemannian Setting

Optimization with Structure:

$$\min_{x \in \mathcal{M}} F(x) = f(x) + h(x),$$

- 
- [HW22b]: Given  $x_0$ ,
- Proximal Gradient 1
  - Proximal Gradient 2
  - Proximal Gradient 3
- $$\left\{ \begin{array}{l} \text{Let } \ell_{x_k}(\eta) = \langle \text{grad} f(x_k), \eta \rangle_{x_k} + \frac{L}{2} \|\eta\|_{x_k}^2 + h(R_{x_k}(\eta)); \\ \eta_k \text{ is a stationary point of } \ell_{x_k} \text{ on } \mathbb{T}_{x_k} \mathcal{M} \text{ and } \ell_{x_k}(0) \geq \ell_{x_k}(\eta_k); \\ x_{k+1} = R_{x_k}(\eta_k); \end{array} \right.$$
- Direction in the tangent space;
  - Well-defined for general manifold;
  - Subproblem is difficult in general (simple for sphere);
  - Any limit point is a critical point;
  - $O(1/k)$  rate for retraction convex  $f$  and  $h$ ;
  - Local convergence rate by Riemannian KL property;

[HW22b] W. Huang and K. Wei. Riemannian proximal gradient methods. *Mathematical Programming*, 194(1-2):371-413, 2022.

# Related Work

## Riemannian Setting

Optimization with Structure:

$$\min_{x \in \mathcal{M}} F(x) = f(x) + h(x),$$

- 
- Proximal Gradient 1
  - Proximal Gradient 2
  - Proximal Gradient 3
- [BJJP25]: Given  $x_0$ ,
- $$\left\{ \begin{array}{l} \text{Let } H_{x_k}(x) = h(x) + \frac{1}{2\lambda} d^2(x, R_{x_k}(-\lambda \text{grad} f(x_k))); \\ x_{k+1} \text{ is a stationary point of } H_{x_k}(x); \\ \text{and } H_{x_k}(x_k) \geq H_{x_k}(x_{k+1}); \end{array} \right.$$

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[BJJP25] R. Bergmann, H. Jasa, P. John, M. Pfeffer. The intrinsic Riemannian proximal gradient method for nonconvex optimization. arXiv:2506.09775, 2025.

# Related Work

## Riemannian Setting

### Optimization with Structure:

$$\min_{x \in \mathcal{M}} F(x) = f(x) + h(x),$$

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- Proximal Gradient 1
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- $$\left\{ \begin{array}{l} \text{Let } H_{x_k}(x) = h(x) + \frac{1}{2\lambda} d^2(x, R_{x_k}(-\lambda \text{grad} f(x_k))); \\ x_{k+1} \text{ is a stationary point of } H_{x_k}(x); \\ \text{and } H_{x_k}(x_k) \geq H_{x_k}(x_{k+1}); \end{array} \right.$$
- $x_{k+1}$  can be viewed as a Riemannian proximal point of  $h$  on manifold;
  - Sublinear convergence rate in nonconvex setting by Exponential map;
  - Any limit point is a critical point by Exponential map;

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[BJJP25] R. Bergmann, H. Jasa, P. John, M. Pfeffer. The intrinsic Riemannian proximal gradient method for nonconvex optimization. arXiv:2506.09775, 2025.

# Riemannian Version of FISTA

Euclidean version:

[BT09] convex: Given  $x_0$ , let  $y_0 = x_0, t_0 = 1$ ;

$$\begin{cases} d_{y_k} = \operatorname{argmin}_p \langle \nabla f(y_k), p \rangle + \frac{L}{2} \|p\|_{\mathbb{F}}^2 + h(y_k + p) \\ x_{k+1} = y_k + d_{y_k} \\ t_{k+1} = \frac{\sqrt{4t_k^2 + 1} + 1}{2} \\ y_{k+1} = x_{k+1} + \frac{t_k - 1}{t_{k+1}} (x_{k+1} - x_k). \end{cases}$$

- 
- Riemannian version 1
  - Riemannian version 2

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- 
- **Riemannian version 1** [HW22a], AManPG: Given  $x_0$ , let  $y_0 = x_0, t_0 = 1$ ;
  - **Riemannian version 2**

$$\begin{cases} \eta_{y_k} = \operatorname{arg min}_{\eta \in T_{y_k}} \mathcal{M} \langle \nabla f(y_k), \eta \rangle + \frac{L}{2} \|\eta\|_F^2 + h(y_k + \eta) \\ x_{k+1} = R_{y_k}(\eta_{y_k}) \\ t_{k+1} = \frac{\sqrt{4t_k^2 + 1} + 1}{2} \\ y_{k+1} = R_{x_{k+1}} \left( \frac{1-t_k}{t_{k+1}} R_{x_{k+1}}^{-1}(x_k) \right). \end{cases}$$

[HW22a] W. Huang and K. Wei. An extension of fast iterative shrinkage-thresholding algorithm to Riemannian optimization for sparse principal component analysis. *Numerical Linear Algebra with Applications*, 29(1): e2409, 2022.

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• Riemannian version 1 [HW22b]: Given  $x_0$ , let  $y_0 = x_0, t_0 = 1$ ;

• Riemannian version 2  $\left\{ \begin{array}{l} \ell_{y_k}(\eta) = \langle \operatorname{grad} f(y_k), \eta \rangle_{y_k} + \frac{L}{2} \|\eta\|_{y_k}^2 + h(R_{y_k}(\eta)) \\ \eta_{y_k} \text{ is a stationary point of } \ell_{y_k} \text{ and } \ell_{y_k}(0) \geq \ell_{y_k}(\eta_{y_k}) \\ x_{k+1} = R_{y_k}(\eta_{y_k}) \\ t_{k+1} = \frac{1 + \sqrt{4t_k^2 + 1}}{2} \\ y_{k+1} = R_{y_k} \left( \frac{t_{k+1} + t_k - 1}{t_{k+1}} \eta_{y_k} - \frac{t_k - 1}{t_{k+1}} R_{y_k}^{-1}(x_k) \right). \end{array} \right.$

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# Riemannian Version of FISTA

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• Riemannian version 1

$$\frac{2}{\theta L} (t_k^2 (F(x_{k+1}) - F(x_*)) - t_{k-1}^2 (F(x_k) - F(x_*))) \leq$$

• Riemannian version 2

$$\| \underbrace{(t_k - 1)R_{y_k}^{-1}(x_k) + R_{y_k}^{-1}(x_*)}_{\hat{W}_k} \|^2 - \| \underbrace{(t_k - 1)R_{y_k}^{-1}(x_k) + R_{y_k}^{-1}(x_*) - t_k \eta y_k}_{\tilde{W}_{k+1}} \|^2$$

- $\hat{W}_k \neq \tilde{W}_k$  in general;
- How to control the difference?

[HW22a] W. Huang and K. Wei. An extension of fast iterative shrinkage-thresholding algorithm to Riemannian optimization for sparse principal component analysis. *Numerical Linear Algebra with Applications*, 29(1): e2409, 2022.

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- 
- Riemannian version 1
  - Riemannian version 2
    - Observe acceleration empirically;
    - No theoretical guarantee for acceleration;

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[HW22b] W. Huang and K. Wei. Riemannian proximal gradient methods. *Mathematical Programming*, 194(1-2):371-413, 2022.

# Related Work

## Riemannian Setting

$$\min_{x \in \mathcal{M}} F(x) = f(x) + h(x),$$

---

In smooth case:  $h = 0$ , [Riemannian Accelerated Gradient Methods](#)  
[LSC<sup>+</sup>17] [ZS18, AS20, JS22] [AOBL21] [MR22] [KY22] [MRP23] [CK25]

- [LSC<sup>+</sup>17]: Computationally hard;
- [ZS18, AS20, JS22]: Strongly convex, close to minimizer;
- [AOBL21]: convex, early stage;
- [MR22]: constant curvature;
- [KY22]: bound constraints;
- [MRP23]: Hadamard manifold;
- [CK25]: Rate independent of curvature;
- [CB22]: Negative curvature obstruct acceleration;

# Related Work

## Riemannian Setting

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[LSC<sup>+</sup>17] [ZS18, AS20, JS22] [AOBL21] [MR22] [KY22] [MRP23] [CK25]

[KY22] Given  $x_0$ , let  $z_0 = x_0$ ;

$$\begin{cases} y_k = \text{Exp}_{x_k} \left( \tau_k \text{Exp}_{x_k}^{-1}(z_k) \right) \\ x_{k+1} = \text{Exp}_{y_k} \left( -\alpha_k \text{grad} f(y_k) \right) \\ v_{y_k} = \beta_k \text{Exp}_{y_k}^{-1}(z_k) - \gamma_k \text{grad} f(y_k) \\ z_{k+1} = \text{Exp}_{x_{k+1}} \left( \Gamma_{y_k}^{x_{k+1}} \left( v_{y_k} - \text{Exp}_{y_k}^{-1}(x_{k+1}) \right) \right). \end{cases}$$

[KY22] J. Kim and I. Yang. Accelerated gradient methods for geodesically convex optimization: tractable algorithms and convergence analysis. PMLR, 162: 11255–11282, 2022.

# Related Work

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- Accelerated convergence rates for geodesically convex  $f$  and geodesically strongly convex  $f$ , respectively;

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# Related Work

## Riemannian Setting

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In smooth case:  $h = 0$ , [Riemannian Accelerated Gradient Methods](#)

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- Accelerated convergence rates for geodesically convex  $f$  and geodesically strongly convex  $f$ , respectively;
- **No unified parameters for accelerated gradient methods that works for both geodesically convex and geodesically strongly convex functions;**

[KY22] J. Kim and I. Yang. Accelerated gradient methods for geodesically convex optimization: tractable algorithms and convergence analysis. PMLR, 162: 11255–11282, 2022.

# Outline

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  - The Proposed Approach
  - Estimation of Convergence Rate
- 3 Adaptive Restart for Riemannian Accelerated Proximal Gradient Method
- 4 Summary

# The Proposed Approach

Riemannian accelerated proximal gradient method (RAPG)

- Riemannian proximal mapping [HW22b];
- Nesterov's acceleration;
- A three-point iterative method;

# The Proposed Approach

Riemannian accelerated proximal gradient method (RAPG)

Initial iterate  $x_0$ , let  $z_0 = x_0$ ;

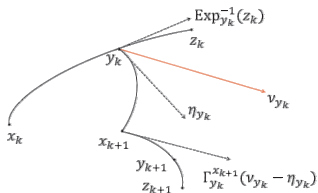
- 1  $y_k = \text{Exp}_{x_k} \left( \tau_k \text{Exp}_{x_k}^{-1}(z_k) \right)$ ;
  - 2  $\eta_{y_k}$  is a stationary point of  $\ell_{y_k}(\eta)$  on  $T_{y_k} \mathcal{M}$  with  $\ell_{y_k}(0) \geq \ell_{y_k}(\eta_{y_k})$ , where  $\ell_{y_k}(\eta) = \langle \text{grad } f(y_k), \eta \rangle + \frac{\theta L}{2} \|\eta\|_{y_k}^2 + h \left( \text{Exp}_{y_k}(\eta) \right)$ ;
  - 3  $x_{k+1} = \text{Exp}_{y_k}(\eta_{y_k})$ ;
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# The Proposed Approach

Riemannian accelerated proximal gradient method (RAPG)

Initial iterate  $x_0$ , let  $z_0 = x_0$ ;

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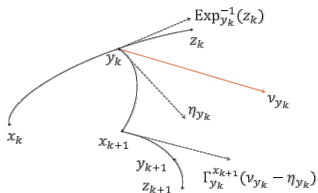
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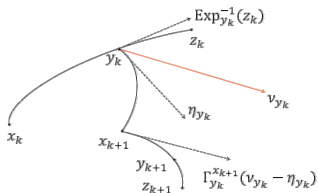
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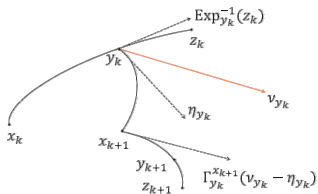
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- ③ Step 3: update  $x_{k+1}$  by exponential map;
- ④ Step 4: update  $z_{k+1}$  by exponential map and parallel transport;

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Riemannian accelerated proximal gradient method (RAPG)

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Next, we will show:

- ① Assumptions on Manifolds and functions;
- ② Parameter expressions for  $\tau_k, \beta_k, \gamma_k$ ;
- ③ Convergence rate of RAPG;

# Assumptions on Manifolds and Functions

Assumption on Manifold:

- 1 Let  $\Omega$  be a geodesically uniquely convex subset of  $\mathcal{M}$ . The diameter of  $\Omega$  satisfies  $\text{diam}(\Omega) \leq D < \infty$ ;
  - 2 The sectional curvature of  $\Omega$  is bounded below by  $\kappa_{\min}$  and above by  $\kappa_{\max}$ . If  $\kappa_{\max} > 0$ , it is additionally assumed that  $D < \frac{\pi}{\sqrt{\kappa_{\max}}}$ ;
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For the eigenvalues of the Hessian matrix of the squared distance function  $\frac{1}{2}d^2(\cdot, p)$  on  $\Omega \subset \mathcal{M}$ , where  $p \in \Omega$ :

- the upper bound:

$$\zeta = \begin{cases} \sqrt{-\kappa_{\min}}D \coth(\sqrt{-\kappa_{\min}}D), & \text{if } \kappa_{\min} < 0 \\ 1, & \text{if } \kappa_{\min} \geq 0 \end{cases}$$

- the lower bound:

$$\delta = \begin{cases} 1, & \text{if } \kappa_{\max} \leq 0 \\ \sqrt{\kappa_{\max}}D \cot(\sqrt{\kappa_{\max}}D), & \text{if } \kappa_{\max} > 0 \end{cases}$$

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Choose  $\xi \geq \zeta$ .

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- 1 The function  $f$  is geodesically  $L$ -smooth and geodesically  $\mu$ -strongly convex ( $\mu \geq 0$ ) in  $\Omega$ ;
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# Assumptions on Manifold and Functions

$\rho$ -retraction-convex:

$\tilde{h}_x(\eta) = h(R_x(\eta)) + \frac{\rho}{2}\|\eta\|^2$  is convex in tangent space.

---

- $\rho > 0$ ,  $h$  is said to be  $\rho$ -weakly retraction-convex with respect to  $R$ ;
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Weakly Retraction-Convex: [A Necessary Assumption](#)

e.g.  $\|x\|_1$  is locally weakly retraction-convex on the embedded submanifold of  $\mathbb{R}^n$ .

# Parameter Expressions for $\beta_k, \gamma_k, \tau_k$

Under assumptions on manifold and functions:

$$A_{k+1} = \frac{\bar{\zeta} + 2\bar{\zeta}A_k + \sqrt{\bar{\zeta}^2 + 4\bar{\zeta}^2 A_k + 4\frac{\mu-\rho}{\theta L-\rho}\bar{\zeta}A_k^2}}{2\left(\bar{\zeta} - \frac{\mu-\rho}{\theta L-\rho}\right)},$$

$$\beta_k = \frac{\bar{\zeta}(\theta L - \rho) + (\mu - \rho)A_k}{\bar{\zeta}(\theta L - \rho) + (\mu - \rho)A_{k+1}}, \gamma_k = \frac{(\theta L - \rho)(A_{k+1} - A_k)}{\bar{\zeta}(\theta L - \rho) + (\mu - \rho)A_{k+1}}, \tau_k = \frac{\beta_k A_{k+1}}{\gamma_k A_k + \beta_k A_{k+1}};$$


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Reduce to Euclidean space:

- if  $\xi = 1, \rho = 0$ , RAPG is FISTA in strongly convex [dST<sup>+</sup>21];
- otherwise, it is new as far as we known;

On manifold:

- Our parameter settings apply to both convex and strongly convex cases on manifold, leading to a **unified accelerated algorithm**.

# Convergence Rate of RAPG

Under assumptions on manifold and functions:

- Sublinear convergence for  $\mu \geq \rho$ :  $O\left(\frac{1}{k^2}\right)$ ;
- Linear convergence for  $\mu > \rho$ :

$$\min \left\{ \left( 1 - \sqrt{\frac{\mu - \rho}{(\theta L - \rho)\xi}} \right)^k C_1, \frac{2}{(k + 2\sqrt{A_0})^2} C_2 \right\}.$$

Assumption on functions:

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$$F(x) = f(x) + h(x)$$

# Convergence Rate of RAPG

## Sketch of the analysis

The core of our analysis is the construction of a potential function (or Lyapunov function)  $\Phi_k$  that combines:

- 1 the function value gap;
- 2 the distance from the iterate to the optimal point; and
- 3 distortion error from curvature;

$$\begin{aligned}\Phi_k = & A_k(F(x_k) - F(x_*)) \\ & + \frac{\xi(\theta L - \rho) + (\mu - \rho)A_k}{2} \left( \left\| \text{Exp}_{x_k}^{-1}(z_k) - \text{Exp}_{x_k}^{-1}(x_*) \right\|^2 \right. \\ & \left. + (\xi - 1) \left\| \text{Exp}_{x_k}^{-1}(z_k) \right\|^2 \right)\end{aligned}$$

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A convergence rate of  $O(1/A_k)$  is achieved if  $\Phi_{k+1} \leq \Phi_k$  is satisfied.

The limit of RAPG:

- RAPG is theoretically supported only under the convexity of both  $f$  and  $h$  on manifolds;
- What happens in the nonconvex case?

We develop an improved version of the method.

# Outline

- 1 Riemannian Proximal Gradient Methods Review
- 2 A Riemannian Accelerated Proximal Gradient Method
- 3 Adaptive Restart for Riemannian Accelerated Proximal Gradient Method
  - Algorithm
  - Convergence Analysis
- 4 Summary

# Adaptive Restart for RAPG

## Adaptive Restart for Riemannian Accelerated Proximal Gradient Method (AR-RAPG)

---

- 1: Set  $z_0 = x_0$ ,  $\tilde{x}_0 = x_0$ ,  $\theta \geq 1$ ,  $L = L_{\text{init}}$ ,  $i = 0$ , and  $j = N_0$ ;
  - 2: for  $k = 0, 1, 2, \dots$  do
  - 3:   if  $k == j$  then
  - 4:      $[\tilde{x}_{i+1}, x_k, z_k, A_k, N_{i+1}, L] = \text{Safeguard}(\tilde{x}_i, x_k, z_k, A_k, N_i, L)$ ;
  - 5:     Set  $j = j + N_{i+1}$  and  $i = i + 1$ ;
  - 6:   end if
  - 7:    $(A_{k+1}, \beta_k, \gamma_k, \tau_k)$  are derived from the same formulas as in RAPG;
  - 8:   Compute  $y_k, x_{k+1}, z_{k+1}$  as in RAPG;
  - 9: end for
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- 

- Safeguard strategy from [HW22b];
- The functions  $f$  and  $h$  are not required to be convex on manifold;
- If the convexity of the functions is not known, we simply set  $\mu = 0$  and  $\rho = 0$ ;

[HW22b] W. Huang and K. Wei. Riemannian proximal gradient methods. *Mathematical Programming*, 194(1-2):371-413, 2022.

# Adaptive Restart for RAPG

## Safeguard

---

Input:  $(\tilde{x}_i, x_k, z_k, A_k, N_i, L)$ ;

Output:  $(\tilde{x}_{i+1}, x_k, z_k, A_k, N_{i+1}, L)$ ;

- 1:  $\eta_{\tilde{x}_i}$  is a stationary point of  $\ell_{\tilde{x}_i}(\eta)$  on  $T_{\tilde{x}_i} \mathcal{M}$  with  $\ell_{\tilde{x}_i}(0) \geq \ell_{\tilde{x}_i}(\eta_{\tilde{x}_i})$ ;
- 2: Set  $\alpha_i = 1$ ,  $i_{ls} = 0$ ;
- 3: while  $F(\text{Exp}_{\tilde{x}_i}(\alpha_i \eta_{\tilde{x}_i})) > F(\tilde{x}_i) - \sigma \alpha_i \|\eta_{\tilde{x}_i}\|^2$  and  $i_{ls} < N_{ls}$  do
  - 4:  $\alpha_i = \rho \alpha_i$ ,  $i_{ls} = i_{ls} + 1$ ;
  - 5: end while
  - 6: if  $i_{ls} == N_{ls}$  then
    - 7:  $L = \tau L$  and go to Step 1; **The estimation of  $L$  is too small**
    - 8: end if
    - 9: if  $F(\text{Exp}_{\tilde{x}_i}(\alpha_i \eta_{\tilde{x}_i})) < F(x_k)$  then
      - 10: **Safeguard takes effect**
      - 11: if  $N_i \neq N_{\max}$  then
        - 12:  $L = \tau L$ ;
        - 13: end if
        - 14:  $x_k = \text{Exp}_{\tilde{x}_i}(\alpha_i \eta_{\tilde{x}_i})$ ,  $z_k = x_k$ ,  $A_k = A_0$ ; {Restart}
        - 15:  $N_{i+1} = \max\{N_i - 1, N_{\min}\}$ ;
        - 16: else
          - 17:  $x_k, z_k$ , and  $A_k$  keep unchanged; **No restart**
          - 18:  $N_{i+1} = \min\{N_i + 1, N_{\max}\}$ ;
          - 19: end if
          - 20:  $\tilde{x}_{i+1} = x_k$ .

---

- Adaptively update the smoothness parameter  $L$ ;
- Guarantee a decrease in the function value after a finite number of iterations;

# Adaptive Restart for RAPG

## Theorem (Convergence)

Under assumptions of Manifolds, if

- 1  $\Omega$  is compact;
- 2 all iterates remain in  $\Omega$ ;
- 3  $f$  is smooth,  $h$  is locally Lipschitz continuous,

then any accumulation point  $\tilde{x}_*$  of the sequence  $\{\tilde{x}_i\}$  generated by AR-RAPG is a stationary point.

# Numerical Experiments

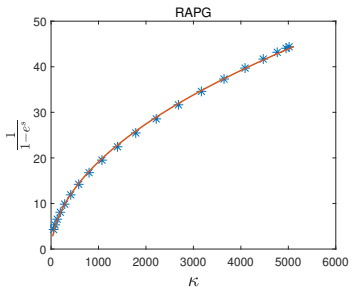
Convergence rate verification of RAPG and RPG

$$\min_{x \in \mathbb{S}^{n-1}} F(x) = \underbrace{-x^T A^T A x}_{f_1(x)} + \underbrace{\lambda \|x\|_1}_{h(x)},$$

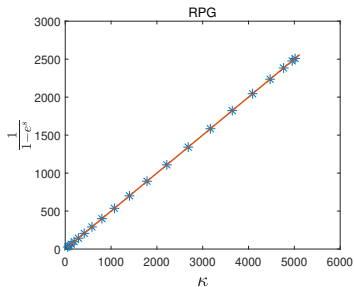
- $A = USV^T + e$ ;
- $S \in \mathbb{R}^{m \times n}$ : first  $m$  columns are  $\text{diag}(m + c, m, m - 1, \dots, 2)$  with  $c$  varying from 0.01 to 1, and the remaining columns are zero;
- $e$  is a small noise;

# Numerical Experiments

Convergence rate verification of RAPG and RPG



(a) RAPG



(b) RPG

**Figure:** Empirical relationship between  $\kappa$  and  $\frac{1}{1-e^s}$  for RAPG and RPG.  
 $m = 20, n = 1000, \lambda = 10^{-4}$ .

# Numerical Experiments

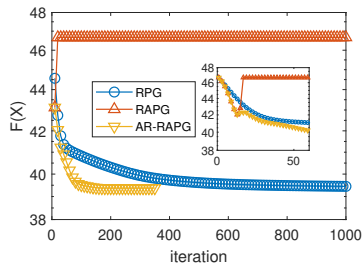
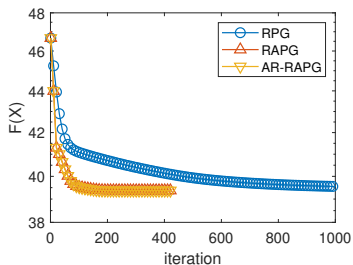
Effectiveness of the safeguard in AR-RAPG

$$\min_{X \in \text{OB}(p,n)} F(X) = \underbrace{\|X^T A^T A X - D^2\|_F^2}_{f_2(X)} + \underbrace{\lambda \|X\|_1}_{h(X)}$$

- Oblique manifold:  
 $\text{OB}(p,n) = \{X \in \mathbb{R}^{n \times p} \mid x_i^T x_i = 1, i = 1, \dots, p\}$ ;
- Entries of  $A$ : standard normal distribution  $\mathcal{N}(0,1)$ ;
- Each column of  $A$ : zero mean and unit 2-norm;

# Numerical Experiments

## Effectiveness of the safeguard in AR-RAPG



**Figure:** Comparison of RPG, RAPG, and AR-RAPG for the SPCA problem on oblique manifold.  $\lambda = 1$ ,  $m = 20$ ,  $n = 200$ ,  $p = 4$ . Left:  $L = 2\|D^2\|_F^2$ ; Right:  $L = 1.2\|D^2\|_F^2$ .

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Sparse PCA problem:

$$\min_{X \in \text{OB}(p,n)} \|X^T A^T A X - D^2\|_F^2 + \lambda \|X\|_1,$$

- 
- $\text{OB}(p,n) = \{X \in \mathbb{R}^{n \times p} \mid x_i^T x_i = 1, i = 1, \dots, p\}$  denotes the oblique manifold;
  - $x_i$  being the  $i$ -th column of  $X$ ;
  - $A \in \mathbb{R}^{m \times n}$  is the data matrix and  $p \leq m$ ;
  - $D$  is a diagonal matrix with the dominant singular values of  $A$  on the diagonal;
- 

Compared with:

- ManPG, ManPG-Ada: in [CMSZ20];
- RPG: in [HW22b];

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Left: the norm of search direction;  
Right: function value.

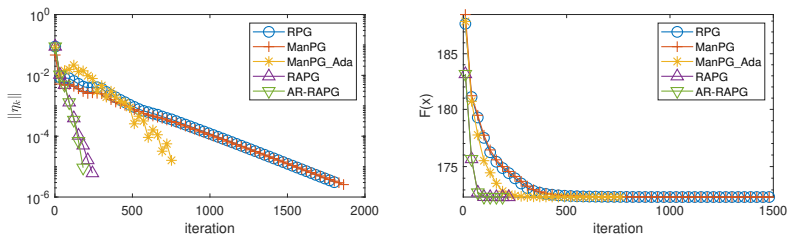


Figure: SPCA problem on oblique manifold.  $n = 200$ ,  $m = 20$ ,  $p = 4$ .

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For  $m = 20$ ,  $p = 4$ ,  $n = \{32, 64, 128, 256\}$ .

Left: number of iterations;

Right: CPU time.

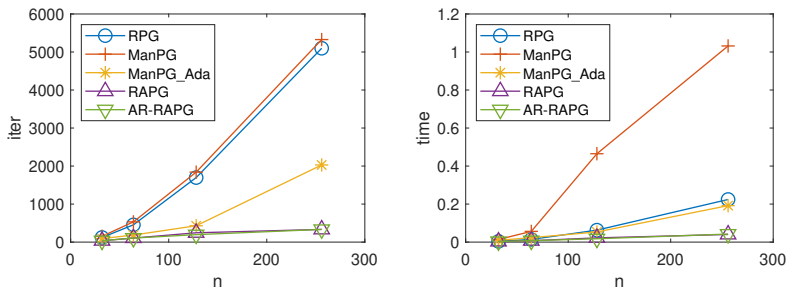


Figure: SPCA problem on oblique manifold.

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For  $m = 20$ ,  $n = 128$ ,  $p = \{1, 2, 3, 4\}$ .

Left: number of iterations;

Right: CPU time.

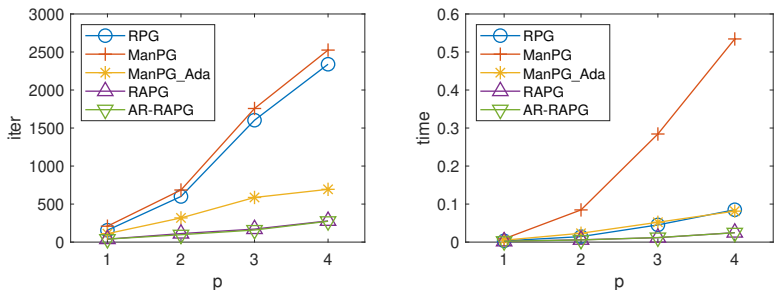


Figure: SPCA problem on oblique manifold.

# Outline

- 1 Riemannian Proximal Gradient Methods Review
- 2 A Riemannian Accelerated Proximal Gradient Method
- 3 Adaptive Restart for Riemannian Accelerated Proximal Gradient Method
- 4 Summary**

# Summary

- Propose a Riemannian accelerated proximal gradient method;
- Accelerated convergence rate is proven;
- Propose an adaptive restart Riemannian accelerated proximal gradient method;
- Global convergence to critical points is proven;
- Numerical experiments show its performance;

ArXiv preprint: S. Feng, Y. Jiang, W. Huang\*, S. Ying, A Riemannian Accelerated Proximal Gradient Method, arXiv:2509.21897, 2025.

Thank you!

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