

A Kaczmarz Method for Blind Deconvolution

Wen Huang

Xiamen University

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Joint work with Zilin Yang

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海南陵水

- Blind deconvolution
- A review of Kaczmarz methods
- A Kaczmarz method for blind deconvolution
- Preliminary numerical results
- Summary and future work

Blind deconvolution

[Blind deconvolution]

Blind deconvolution is to recover two unknown signals from their convolution.

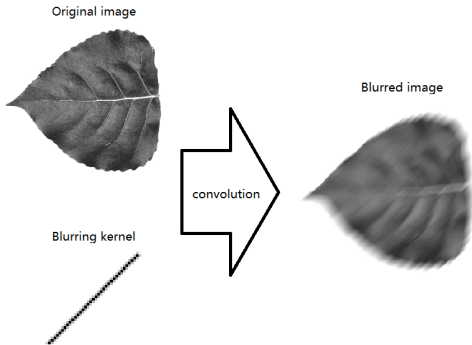
Blurred image



Blind deconvolution

[Blind deconvolution]

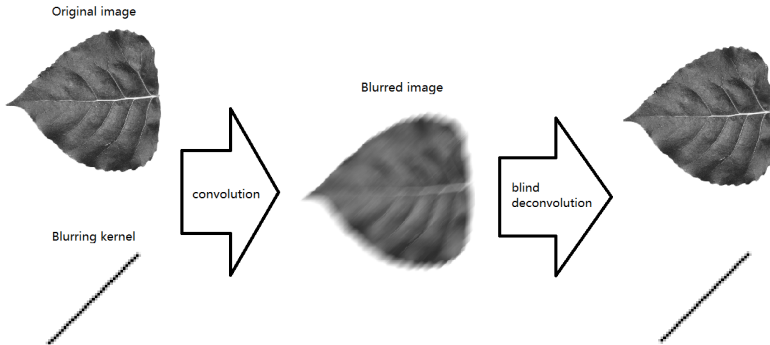
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Blind deconvolution

[Blind deconvolution]

Blind deconvolution is to recover two unknown signals from their convolution.



Blind deconvolution

[Blind deconvolution (Discretized version)]

Blind deconvolution is to recover two unknown signals $\mathbf{w} \in \mathbb{C}^L$ and $\mathbf{x} \in \mathbb{C}^L$ from their convolution $\mathbf{y} = \mathbf{w} * \mathbf{x} \in \mathbb{C}^L$.

- We only consider circular convolution:

$$\begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \mathbf{y}_3 \\ \vdots \\ \mathbf{y}_L \end{bmatrix} = \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_L & \mathbf{w}_{L-1} & \dots & \mathbf{w}_2 \\ \mathbf{w}_2 & \mathbf{w}_1 & \mathbf{w}_L & \dots & \mathbf{w}_3 \\ \mathbf{w}_3 & \mathbf{w}_2 & \mathbf{w}_1 & \dots & \mathbf{w}_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{w}_L & \mathbf{w}_{L-1} & \mathbf{w}_{L-2} & \dots & \mathbf{w}_1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \\ \vdots \\ \mathbf{x}_L \end{bmatrix}$$

- Let $y = \mathbf{F}\mathbf{y}$, $w = \mathbf{F}\mathbf{w}$, and $x = \mathbf{F}\mathbf{x}$, where \mathbf{F} is the DFT matrix;
- $y = w \odot x$, where \odot is the Hadamard product, i.e., $y_i = w_i x_i$.
- **Equivalent question:** Given y , find w and x .

Blind deconvolution

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 - Leads to mathematical rigor; ($L/(K + N)$ reasonably large)

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Problem under the assumption

Given $y \in \mathbb{C}^L$, $B \in \mathbb{C}^{L \times K}$ and $C \in \mathbb{C}^{L \times N}$, find $h \in \mathbb{C}^K$ and $m \in \mathbb{C}^N$ so that

$$y = Bh \odot \overline{Cm} = \text{diag}(Bhm^* C^*).$$

$$\text{Find } h, m, \text{ s. t. } y = \text{diag}(Bhm^* C^*);$$

- Ahmed et al. [ARR14] ¹
 - Convex relaxation ($X = hm^*$): $\min_{y = \text{diag}(BXC^*)} \|X\|_n$;
 - The unique minimizer high probability true solution;
 - The convex problem may be expensive to solve;

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Blind deconvolution

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- Li et al. [LLSW19] ²

- Nonconvex problem: $\min_{(h,m) \in \mathbb{C}^K \times \mathbb{C}^N} \|y - \text{diag}(Bhm^* C^*)\|_2^2$;
- (Wirtinger flow + good initialization) $\xrightarrow{\text{high probability}}$ true solution;
- Lower successful recovery rate than alternating minimization;

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³Wen Huang, Paul Hand. Blind Deconvolution by a Steepest Descent Algorithm on a Quotient Manifold, *SIAM Journal on Imaging Sciences*, 11:4, pp. 2757-2785, 2018.

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The theoretical results assume (1) C : complex Gaussian distribution
and (2) $B^* B = I_K$ and $\|b_i\|^2 < \phi K/L, i = 1, \dots, L$.

Blind deconvolution

Problem

Given $y \in \mathbb{C}^L$, $B \in \mathbb{C}^{L \times K}$ and $C \in \mathbb{C}^{L \times N}$, find $h \in \mathbb{C}^K$ and $m \in \mathbb{C}^N$ so that

$$y = Bh \odot \overline{Cm} = \text{diag}(Bhm^* C^*).$$

Kaczmarz method

It is an iterative projection technique by repeatedly projecting a current estimate onto a **space** that contains the **solution set**.

- **Solution set:**

$$\mathcal{N} = \{X \in \mathbb{C}^{K \times N} : \text{rank}(X) = 1, y = \text{diag}(BXC^*)\};$$

- **Space:**

$$\mathcal{N}_i = \{X \in \mathbb{C}^{K \times N} : \text{rank}(X) = 1, y_i = b_i^* X c_i\}, i = 1, 2, \dots, L.$$

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A review of Kaczmarz methods

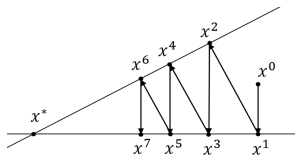
Linear system and phase retrieval

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Kaczmarz method for the linear system $y = Ax$

- $\mathcal{N}_i = \{x \mid a_i^T x = y_i\}$ is a hyperplane;
- Projection to a hyperplane has a closed-form solution;



A review of Kaczmarz methods

Linear system and phase retrieval

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Kaczmarz method for the linear system $y = Ax$

Input: y, A , an initial point x_0 ;

1: **for** $l = 0, 1, 2, \dots$,

2: Select a row of A , denoted by a_i^T ;

3: Compute $x_{l+1} = x_l + \frac{y_i - a_i^T x_l}{\|a_i\|_2^2} a_i$;

4: **end for**

A review of Kaczmarz methods

Linear system and phase retrieval

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Kaczmarz method for the linear system $y = Ax$

- Originally for linear equations [Kac37];
- Sweeps through the rows cyclically [DH97, Gal05, XZ02];
- Selects i row with probability $\frac{\|a_i\|_2^2}{\|A\|_F^2}$ and converges linearly [SV09];

A review of Kaczmarz methods

Linear system and phase retrieval

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Kaczmarz method for the phase retrieval $y = |Ax|^2$

- $\mathcal{N}_i = \{x \mid |a_i^* x|^2 = y_i\}$ is a set of hyperplanes;
- Projection to a union of hyperplanes has a closed-form solution;

A review of Kaczmarz methods

Linear system and phase retrieval

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It is an iterative projection technique by repeatedly projecting a current estimate onto a **space** that contains the **solution set**.

Kaczmarz method for phase retrieval $y = |Ax|^2$

Input: y, A , an initial point x_0 ;

1: **for** $l = 0, 1, 2, \dots$,

2: Select a row of A , denoted by a_l^T ;

3: Compute $\theta_l = \angle \langle a_l, x_l \rangle$;

4: Compute $x_{l+1} = x_l + \frac{\sqrt{y_l} e^{\sqrt{-1}\theta_l} - \langle a_l, x_l \rangle}{\|a_l\|_2^2} a_l$;

5: **end for**

A review of Kaczmarz methods

Linear system and phase retrieval

Kaczmarz method

It is an iterative projection technique by repeatedly projecting a current estimate onto a **space** that contains the **solution set**.

Kaczmarz method for phase retrieval $y = |Ax|^2$

- Extended for phase retrieval problem [Wei15, LGL15];
- An increasing number of mathematicians have contributed to this area of research [JG17, TV19, RJ20, RFK21, ZF22, HW22];
- Linearly converge to a neighborhood [Wei15] and linearly converge to the solution [HW22] under restricted strong convexity;

A review of Kaczmarz methods

General framework

A general setting of Kaczmarz methods

Input: An initial point x_0 ;

- 1: **for** $l = 0, 1, 2, \dots$, **do**
 - 2: Select a set \mathcal{N}_l that contains the solution space;
 - 3: Compute $x_{l+1} = \arg \min_{x \in \mathcal{N}_l} \|x - x_l\|^2$;
 - 4: **end for**
-

Linear system

- \mathcal{N}_l : a hyperplane $a^T x = b_l$;
- Closed-form solution exists;

Phase retrieval

- \mathcal{N}_l : a union of hyperplanes
- Closed-form solution exists;

A review of Kaczmarz methods

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Blind deconvolution

- $\mathcal{N}_l = \{X \in \mathbb{C}^{K \times N} : \text{rank}(X) = 1, y_l = b_l^* X c_l\}$;
- Projection: $X_{l+1} = \arg \min_{X \in \mathcal{N}_l} \|X - X_l\|$;

A review of Kaczmarz methods

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Does the projection yield a closed form solution?

A review of Kaczmarz methods

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Blind deconvolution

- $\mathcal{N}_l = \{X \in \mathbb{C}^{K \times N} : \text{rank}(X) = 1, y_l = b_l^* X c_l\}$;
- **Projection:** $X_{l+1} = \arg \min_{X \in \mathcal{N}_l} \|X - X_l\|$;

Does the projection yield a closed form solution?

No, as far as we know

- Blind deconvolution
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A Kaczmarz method for blind deconvolution

Computation of the projection

A Kaczmarz method for blind deconvolution

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Key question: can we compute the projection efficiently?

¹Yixian Liu, An efficient method for non-convex blind deconvolution, *IEEE Access*, 7:113663–113674, 2019

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- 2: Select $\mathcal{N}_l = \{X \in \mathbb{C}^{K \times N} : \text{rank}(X) = 1, y_l = b_l^* X c_l\}$;
- 3: **Compute** $X_{l+1} \approx \arg \min_{X \in \mathcal{N}_l} \|X - X_l\|^2$;
- 4: **end for**

Key question: can we compute the projection efficiently?

An existing Kaczmarz method ([Liu19]¹)

- Nonconvex problem: $\min_{(h,m) \in \mathbb{C}^K \times \mathbb{C}^N} \sum_{i=1}^L |y_i - b_i^* h m^* c_i|^2$;
- Alternating minimization for one round;
- Good initialization $\xrightarrow{\text{high probability}}$ true solution;

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Key question: can we compute the projection efficiently?

The proposed Kaczmarz method

- Equivalently reduce to 2×2 subproblem;
- Multiple approaches for the reduced problem;

A Kaczmarz method for blind deconvolution

Computation of the projection

Equivalently reduce to 2×2 subproblem;

Assumption

$y_i \neq 0, i = 1, \dots, L.$

Theorem

Suppose the above assumption holds. Then the two problems

$$\min_{X \in \mathbb{C}^{K \times N}, \text{rank}(X)=1, y_i = b_i^* X c_i} \|X - h_l m_l^*\|_F^2 \text{ and } \min_{\tilde{X} \in \mathbb{C}^{2 \times 2}, \text{rank}(\tilde{X})=1, y_i = \tilde{b}_i^* \tilde{X} \tilde{c}_i} \|\tilde{X} - \tilde{h}_l \tilde{m}_l^*\|_F^2$$

are equivalent in the sense that

$$X_* : \text{stationary/local/global} \iff \tilde{X}_* = Q_1^* X_* Q_2 : \text{stationary/local/global};$$

- QR decomposition: $[h_l \ b_i] = Q_1 R_1$ and $[m_l \ c_i] = Q_2 R_2$;
- $\tilde{h}_l = Q_1^* h_l$, $\tilde{m}_l = Q_2^* m_l$, $\tilde{b}_i = Q_1^* b_i$, and $\tilde{c}_i = Q_2^* c_i$;

A Kaczmarz method for blind deconvolution

Computation of the projection

Equivalently reduce to 2×2 subproblem;

Assumption

$$y_i \neq 0, i = 1, \dots, L.$$

- Easy to verify;
- $y_i = 0 \implies b_i^* h_i = 0$ or $c_i^* m_i = 0$
 $\implies b_i = 0$ or $c_i = 0 \implies i$ -th element removable.
- $\mathbb{C}_1^{K \times N}$ and \mathcal{H}_i intersect transversally $\iff \mathcal{N}_i = \mathbb{C}_1^{K \times N} \cap \mathcal{H}_i$ is an embedded submanifold;
 - $T_X \mathcal{N}_i = T_X(\mathbb{C}_1^{K \times N} \cap \mathcal{H}_i) = T_X \mathbb{C}_1^{K \times N} \cap T_X \mathcal{H}_i$;
 - $N_X \mathcal{N}_i = N_X(\mathbb{C}_1^{K \times N} \cap \mathcal{H}_i) = N_X \mathbb{C}_1^{K \times N} + N_X \mathcal{H}_i$;

A Kaczmarz method for blind deconvolution

Computation of the projection

Theorem

Suppose the above assumption holds. Then the two problems

$$\min_{X \in \mathbb{C}^{K \times N}, \text{rank}(X)=1, y_i = b_i^* X c_i} \|X - h_l m_l^*\|_F^2 \quad \text{and} \quad \min_{\tilde{X} \in \mathbb{C}^{2 \times 2}, \text{rank}(\tilde{X})=1, y_i = \tilde{b}_i^* \tilde{X} \tilde{c}_i} \|\tilde{X} - \tilde{h}_l \tilde{m}_l^*\|_F^2$$

are equivalent in the sense that

X_* : stationary/local/global $\iff \tilde{X}_* = Q_1^* X_* Q_2$: stationary/local/global;
where $[h_l \ b_i] = Q_1 R_1$ and $[m_l \ c_i] = Q_2 R_2$, $[\tilde{h}_l \ \tilde{c}_i] = Q_1^* [h_l \ c_i]$,
 $\tilde{m}_l = Q_2^* m_l$, $\tilde{b}_i = Q_1^* b_i$, and $\tilde{c}_i = Q_2^* c_i$;

A Sketch of Proof

1. Let $h_* m_*^*$ denote X_* . Then $h_* \in \text{span}(Q_1)$ and $m_* \in \text{span}(Q_2)$;
2. Equivalence of the first-order optimality conditions (manifold structures);
3. Equivalence of local and global minimizers;

A Kaczmarz method for blind deconvolution

Computation of the projection

Solve

$$\min_{X \in \mathbb{C}^{K \times N}, \text{rank}(X)=1, y_i = b_i^* X c_i} \|X - h_l m_l^*\|_F^2$$

by

$$\min_{\tilde{X} \in \mathbb{C}^{2 \times 2}, \text{rank}(\tilde{X})=1, y_i = \tilde{b}_i^* \tilde{X} \tilde{c}_i} \|\tilde{X} - \tilde{h}_l \tilde{m}_l^*\|_F^2$$

Algorithm: Compute $X_{l+1} = \arg \min_{X \in \mathbb{C}^{K \times N}, \text{rank}(X)=1, y_i = b_i^* X c_i} \|X - h_l m_l^*\|_F^2$

Input: y_i, b_i, c_i, h_l, m_l ;

- 1: Compute QR decomposition $Q_1 R_1 = [h_l \quad b_i]$, $Q_2 R_2 = [m_l \quad c_i]$;
 - 2: Compute $\tilde{h}_l \leftarrow Q_1^* h_l = R_1(:, 1)$, $\tilde{m}_l \leftarrow Q_2^* m_l = R_2(:, 1)$,
 $\tilde{b}_i \leftarrow Q_1^* b_i = R_1(:, 2)$, $\tilde{c}_i \leftarrow Q_2^* c_i = R_2(:, 2)$;
 - 3: Find $(\tilde{h}_{l+1}, \tilde{m}_{l+1}) = \underset{y_i = \tilde{b}_i^* h m^* \tilde{c}_i}{\operatorname{argmin}} \|h m^* - \tilde{h}_l \tilde{m}_l^*\|_F^2$;
 - 4: Compute $h_{l+1} \leftarrow Q_1 \tilde{h}_{l+1}$, $m_{l+1} \leftarrow Q_2 \tilde{m}_{l+1}$;
-

A Kaczmarz method for blind deconvolution

Computation of the projection

Solve

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by

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 - 2: Compute $\tilde{h}_l \leftarrow Q_1^* h_l = R_1(:, 1)$, $\tilde{m}_l \leftarrow Q_2^* m_l = R_2(:, 1)$,
 $\tilde{b}_i \leftarrow Q_1^* b_i = R_1(:, 2)$, $\tilde{c}_i \leftarrow Q_2^* c_i = R_2(:, 2)$;
 - 3: Find $(\tilde{h}_{l+1}, \tilde{m}_{l+1}) = \underset{y_i = \tilde{b}_i^* h m^* \tilde{c}_i}{\operatorname{argmin}} \|h m^* - \tilde{h}_l \tilde{m}_l^*\|_F^2$;
 - 4: Compute $h_{l+1} \leftarrow Q_1 \tilde{h}_{l+1}$, $m_{l+1} \leftarrow Q_2 \tilde{m}_{l+1}$;
-

How to solve this small-dimensional problem?

A Kaczmarz method for blind deconvolution

Computation of the projection

We have tried three approaches

- Find all KKT points and find the global minimizer;
 - QCQP formulation and find the global solution (Gurobi);
 - Riemannian optimization formulation;
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All stationary points hm^* satisfy that $\exists \alpha, \beta \in \mathbb{C}$ such that

$$\underbrace{\tilde{h}\tilde{m}^* - \tilde{h}_l\tilde{m}_l}_{\text{Euclidean gradient}} = \underbrace{\alpha\tilde{h}_\perp\tilde{m}_\perp^* + \beta\tilde{b}_i\tilde{c}_i^*}_{\text{Normal space}} \implies$$

$$\beta\tilde{b}_i\tilde{c}_i^* + \tilde{h}_l\tilde{m}_l = \tilde{h}\tilde{m}^* - \alpha\tilde{h}_\perp\tilde{m}_\perp^* \implies$$

given β , then \tilde{h} and \tilde{m} can be found.

View \tilde{h} and \tilde{m} as a function of β . Use $y_i = \tilde{b}^*\tilde{h}\tilde{m}^*\tilde{c}_i$ to find β .

A Kaczmarz method for blind deconvolution

Computation of the projection

We have tried three approaches

- Find all KKT points and find the global minimizer;
- QCQP formulation and find the global solution (Gurobi);
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View \tilde{h} and \tilde{m} as a function of β . Use $y_i = \tilde{b}^*\tilde{h}\tilde{m}^*\tilde{c}_i$ to find β .

Too complicated

A Kaczmarz method for blind deconvolution

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QCQP formulation

$$\begin{aligned} \min_{(u,v)} \quad & \|\bar{h}\bar{m}^* + \tilde{b}_{i,\perp}u^* + v\tilde{c}_{i,\perp}^* - \tilde{h}_l\tilde{m}_l^*\|_F^2 \\ \text{subject to} \quad & \det(\bar{h}\bar{m}^* + \tilde{b}_{i,\perp}u^* + v\tilde{c}_{i,\perp}^*) = 0, \end{aligned}$$

where $\bar{h}\bar{m}^*$ denotes a particular solution to the equation $y_i = \tilde{b}_i^* h m^* \tilde{c}_i$.

A Kaczmarz method for blind deconvolution

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where $\bar{h}\bar{m}^*$ denotes a particular solution to the equation $y_i = \tilde{b}_i^* h m^* \tilde{c}_i$.

Find a global minimizer

A Kaczmarz method for blind deconvolution

Computation of the projection

We have tried three approaches

- Find all KKT points and find the global minimizer;
 - QCQP formulation and find the global solution (Gurobi);
 - **Riemannian optimization formulation;**
-

Riemannian optimization formulation

$$\min \|\tilde{X} - \tilde{h}_i \tilde{m}_i^*\|_F^2, \text{ subject to } \tilde{X} \in \mathcal{N}_i,$$

where $\mathcal{N}_i = \{\tilde{X} \in \mathbb{C}^{2 \times 2} \mid \text{rank}(\tilde{X}) = 1, y_i = \tilde{b}_i^* \tilde{X} \tilde{c}_i\}$.

A Kaczmarz method for blind deconvolution

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Most efficient in our experiments

Not guarantee to find a global minimizer

A Kaczmarz method for blind deconvolution

Computation of the projection

Riemannian optimization formulation

$$\min \|\tilde{X} - \tilde{h}_i \tilde{m}_i^*\|_F^2, \text{ subject to } \tilde{X} \in \mathcal{N}_i,$$

where $\mathcal{N}_i = \{\tilde{X} \in \mathbb{C}^{2 \times 2} \mid \text{rank}(\tilde{X}) = 1, y_i = \tilde{b}_i^* \tilde{X} \tilde{c}_i\}$.

Manifold of \mathcal{N}_i :

- $T_{\tilde{X}} \mathcal{N}_i = \{\sigma \tilde{h} \tilde{m}^* + h_p \tilde{m}^* + \tilde{h} m_p^*\} \cap \{\tilde{X} \in \mathbb{C}^{2 \times 2} : \mathbf{0} = \tilde{b}_i^* \tilde{X} \tilde{c}_i\}$
 $= \{[\tilde{h} \quad \tilde{h}_\perp] \begin{bmatrix} \sigma & \alpha \\ \beta & 0 \end{bmatrix} \begin{bmatrix} \tilde{m}^* \\ \tilde{m}_\perp^* \end{bmatrix}\} \cap \{\tilde{X} \in \mathbb{C}^{2 \times 2} : \mathbf{0} = \tilde{b}_i^* \tilde{X} \tilde{c}_i\};$
- $N_{\tilde{X}} \mathcal{N}_i = \{\alpha \tilde{h}_\perp \tilde{m}_\perp^* + \beta \tilde{b}_i \tilde{c}_i^*\} = \text{span}\{\tilde{h}_\perp \tilde{m}_\perp^*, \tilde{b}_i \tilde{c}_i^*\};$
- $\text{Proj}_{\tilde{X}}(\mathbf{G}) = (I - \mathbf{Q}\mathbf{Q}^*)\mathbf{G}$, where \mathbf{Q} is an orthonormal basis of $N_{\tilde{X}} \mathcal{N}_i$;
- $\text{grad } f(\tilde{X}) = \text{Proj}_{\tilde{X}}(\nabla f(\tilde{X}))$, where $\nabla f(\tilde{X}) = 2(\tilde{X} - \tilde{h}_i \tilde{m}_i^*)$;
- $R(W) = \frac{\pi_1(W)}{\tilde{b}_i^* \pi_1(W) \tilde{c}_i} y_i$, where $\pi_1(W) = \sigma_1 u_1 v_1^*$;

A Kaczmarz method for blind deconvolution

Computation of the projection

- Initial point Z_0
 - projection $\tilde{X}_l = \tilde{h}_l \tilde{m}_l^*$ to the hyperplane gets $\tilde{X}_l + \alpha \tilde{b}_l \tilde{c}_l^*$, where
$$\alpha = \frac{y_l - \tilde{b}_l^* \tilde{X}_l \tilde{c}_l}{\|\tilde{b}_l^*\|_2^2 \|\tilde{c}_l\|_2^2};$$
 - $Z_0 = R(\tilde{X}_l + \alpha \tilde{b}_l \tilde{c}_l^*)$, i.e., truncated SVD for $\tilde{X}_l + \alpha \tilde{b}_l \tilde{c}_l^*$ and then rescale the resulting rank one;

A Kaczmarz method for blind deconvolution

Convergence analysis

Convergent results in existing blind deconvolution [ARR14, LLSW19, HH18]

- Good initialization $\xrightarrow{\text{high probability}}$ true solution;
 - Conditions on B and C
 - C : complex Gaussian distribution;
 - $B^*B = I_K$ and $\|b_i\|^2 < \phi K/L, i = 1, \dots, L$;
-

Convergent results in existing Kaczmarz methods

- Linear system: converges linearly [SV09];
- Phase retrieval: converges linearly to a neighborhood of the solution [Wei15];
- Weak conditions on the matrices (solution exists);

A Kaczmarz method for blind deconvolution

Convergence analysis

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A Kaczmarz method for blind deconvolution

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- Phase retrieval: converges linearly to a neighborhood of the solution [Wei15];
- **Weak conditions on the matrices (solution exists);**

local convergence analysis with weak conditions

A Kaczmarz method for blind deconvolution

Convergence analysis

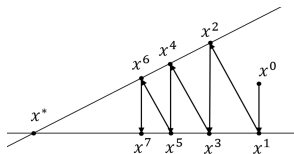


Figure 1: Diagram of Kaczmarz method for a linear system.

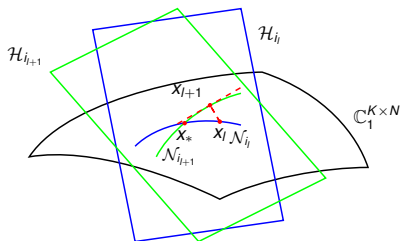


Figure 2: Diagram of Kaczmarz method for Blind deconvolution at l -th step.

The curvature near x_* plays an important role.

A Kaczmarz method for blind deconvolution

Convergence analysis

Lemma

Let \mathcal{N}_i denote the manifold $\{X \in \mathbb{C}^{K \times N} \mid y = b_i^* X c_i, \text{rank}(X) = 1\}$. For any $\delta > 0$, there exists $K_{\delta,i} > 0$ such that for any $\eta_X \in T_X \mathcal{N}_i$ with $\|\eta_X\| = 1$, it holds that the geodesic γ_X satisfies

$$\gamma_X(t\eta_X) = X + t\eta_X + t^2 G(X, t, \eta_X), \forall |t| < \delta$$

and $\|G(X, t, \eta_X)\| \leq K_{\delta,i}$, where $G(X, t, \eta_X) \in N_X \mathcal{N}_i$.

- a straight line + higher-order term;
- $K_{\delta,i}$ depends on the curvature of \mathcal{N}_i near X ;
- $K_{\delta,i} = 0$ in a Euclidean space;

A Kaczmarz method for blind deconvolution

Convergence analysis

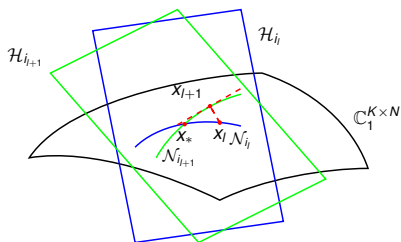


Figure 3: Diagram of Kaczmarz method for Blind deconvolution at l -th step.

$$\mathcal{N}_i \text{ from } X_* \text{ to } X_l: \quad \gamma_{X_*}(t\eta) = X_* + t\eta + t^2 G(X_*, t, \eta)$$

$$\mathcal{N}_{i+1}: \text{ along } \xi = P_{T_{X_*} \mathcal{N}_{i+1}} \eta: \quad \gamma_{X_*}(t\xi) = X_* + t\xi + t^2 G(X_*, t, \xi)$$

where $\eta \in T_{X_*} \mathcal{N}_i$, $\xi \in T_{X_*} \mathcal{N}_{i+1}$, $G(X_*, t, \eta) \in N_{X_*} \mathcal{N}_i$,
 $G(X_*, t, \xi) \in N_{X_*} \mathcal{N}_{i+1}$, $\|G(X_*, t, \eta)\| \leq K_1$ and $\|G(X_*, t, \xi)\| \leq K_2$.

A Kaczmarz method for blind deconvolution

Convergence analysis

Theorem

For a sufficiently small $\delta > 0$, there exists $\rho > 0$ such that for any $\alpha \leq \rho$, it holds that

$$\|X_{l+1} - X_*\| \leq \frac{\langle \eta, \xi \rangle + \delta + \alpha(\langle \eta, \xi \rangle + \delta)^2 K}{1 - \alpha K} \|X_l - X_*\|,$$

where $K = \max\{K_1, K_2\}$.

- Blind deconvolution
- A review of Kaczmarz methods
- A Kaczmarz method for blind deconvolution
- Preliminary numerical results
- Summary and future work

Numerical results

Problem Setting

- B and C are both Gaussian random matrices;
- h_* and m_* are Gaussian random vectors;
- y are noiseless, i.e., $y = Bh_* \odot \overline{Cm_*}$;
- stopping criterion $\|y - Bh \odot \overline{Cm}\|_2 / \|y\| \leq 10^{-4}$;

Numerical results

Efficiency

- $K = N = 100$;
- $RMSE$ denotes the relative error $\frac{\|hm^T - h_*m_*^T\|}{\|h_*\| \|m_*\|}$;

	$L = 500$				
Algorithms	NCBT	NCBB	AMA	ROBB	RKM
nBh/nCm	225	79	157	83	60
$RMSE$	5.74_{-4}	5.57_{-4}	5.73_{-4}	5.71_{-4}	2.09_{-4}
	$L = 600$				
Algorithms	NCBT	NCBB	AMA	ROBB	RKM
nBh/nCm	137	66	109	59	33
$RMSE$	4.82_{-4}	1.78_{-4}	4.26_{-4}	4.65_{-4}	1.86_{-4}

- NCBT: gradient descent [LLSW19];
- NCBB: [LLSW19] with BB step size;
- AMA: alternating minimization in [HH18];
- ROBB: Riemannian SD in [HH18];
- RKM: proposed one;

- Blind deconvolution
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Summary

- Introduce blind deconvolution;
 - Review Kaczmarz methods for linear system, phase retrieval, and blind deconvolution;
 - Propose a Kaczmarz method for blind deconvolution;
 - Local convergence analysis;
 - Preliminary numerical results;
-

Future work

- Convergence analysis;
- Block Kaczmarz method;
- Systematic numerical experiments;

Thank you!

A block Kaczmarz method for blind deconvolution

Computation of the projection

A block Kaczmarz method for blind deconvolution

Input: An initial point X_0 , partition $T = \{\Gamma_1, \dots, \Gamma_{N_b}\}$ of the row indices $\{1, \dots, L\}$;

1: **for** $l = 0, 1, 2, \dots$, **do**

2: Select a block Γ_i and get

$$\mathcal{N}_{\Gamma_i} = \{X \in \mathbb{C}^{K \times N} : \text{rank}(X) = 1, y_{\Gamma_i} = b_{\Gamma_i}^* X c_{\Gamma_i}\};$$

3: Compute $X_{l+1} = \arg \min_{X \in \mathcal{N}_{\Gamma_i}} \|X - X_l\|^2$;

4: **end for**

Equivalently reduce to $(1 + \frac{L}{N_b}) \times (1 + \frac{L}{N_b})$ subproblem;

A block Kaczmarz method for blind deconvolution

Computation of the projection

Equivalently reduce to $(1 + \frac{L}{N_b}) \times (1 + \frac{L}{N_b})$ subproblem;

Solve

$$\min_{X \in \mathbb{C}^{K \times N}, \text{rank}(X)=1, y_{\Gamma_i} = b_{\Gamma_i}^* X c_{\Gamma_i}} \|X - h_l m_l^*\|_F^2$$

by

$$\min_{\tilde{X} \in \mathbb{C}^{(1+\frac{L}{N_b}) \times (1+\frac{L}{N_b})}, \text{rank}(\tilde{X})=1, y_{\Gamma_i} = \tilde{b}_{\Gamma_i}^* \tilde{X} \tilde{c}_{\Gamma_i}} \|\tilde{X} - \tilde{h}_l \tilde{m}_l^*\|_F^2$$

- QR decomposition: $[h_l \ b_{\Gamma_i}] = Q_1 R_1$ and $[m_l \ c_{\Gamma_i}] = Q_2 R_2$;
- $\tilde{h}_l = Q_1^* h_l$, $\tilde{m}_l = Q_2^* m_l$, $\tilde{b}_{\Gamma_i} = Q_1^* b_{\Gamma_i}$, and $\tilde{c}_{\Gamma_i} = Q_2^* c_{\Gamma_i}$;
- $\tilde{h}_l, \tilde{m}_l \in \mathbb{C}^{(1+\frac{L}{N_b}) \times 1}$, $\tilde{b}_{\Gamma_i}, \tilde{c}_{\Gamma_i} \in \mathbb{C}^{(1+\frac{L}{N_b}) \times \frac{L}{N_b}}$

A block Kaczmarz method for blind deconvolution

Computation of the projection

Solve

$$\min_{X \in \mathbb{C}^{K \times N}, \text{rank}(X)=1, y_{\Gamma_i} = b_{\Gamma_i}^* X c_{\Gamma_i}} \|X - h_l m_l^*\|_F^2$$

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Algorithm: Compute $X_{l+1} = \arg \min_{X \in \mathbb{C}^{K \times N}, \text{rank}(X)=1, y_{\Gamma_i} = b_{\Gamma_i}^* X c_{\Gamma_i}} \|X - h_l m_l^*\|_F^2$

Input: $y_{\Gamma_i}, b_{\Gamma_i}, c_{\Gamma_i}, h_l, m_l$;

1: Compute QR decomposition

$$Q_1 R_1 = [h_l \quad b_{\Gamma_i}], \quad Q_2 R_2 = [m_l \quad c_{\Gamma_i}];$$

2: Compute $\tilde{h}_l \leftarrow Q_1^* h_l = R_1(:, 1), \tilde{m}_l \leftarrow Q_2^* m_l = R_2(:, 1),$

$$\tilde{b}_{\Gamma_i} \leftarrow Q_1^* b_{\Gamma_i} = R_1(:, 2 : \text{end}), \tilde{c}_{\Gamma_i} \leftarrow Q_2^* c_{\Gamma_i} = R_2(:, 2 : \text{end});$$

3: Find $(\tilde{h}_{l+1}, \tilde{m}_{l+1}) = \underset{y_{\Gamma_i} = \tilde{b}_{\Gamma_i}^* h m^* \tilde{c}_{\Gamma_i}}{\text{argmin}} \|h m^* - \tilde{h}_l \tilde{m}_l^*\|_F^2;$

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A block Kaczmarz method for blind deconvolution

Computation of the projection

Solve

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by

$$\min_{\tilde{X} \in \mathbb{C}^{(1+\frac{L}{N_b}) \times (1+\frac{L}{N_b})}, \text{rank}(\tilde{X})=1, y_{\Gamma_i} = \tilde{b}_{\Gamma_i}^* \tilde{X} \tilde{c}_{\Gamma_i}} \|\tilde{X} - \tilde{h}_l \tilde{m}_l^*\|_F^2$$

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A block Kaczmarz method for blind deconvolution

Computation of the projection

We have tried two approaches

- QCQP formulation and find the global solution (Gurobi);
 - Riemannian optimization formulation;
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QCQP formulation

$$\begin{aligned} \min_{(u,v)} \quad & \|\bar{h}\bar{m}^* + \tilde{b}_{\Gamma_i,\perp} u^* + v\tilde{c}_{\Gamma_i,\perp}^* - \tilde{h}_l\tilde{m}_l^*\|_F^2 \\ \text{subject to} \quad & \text{rank}(\bar{h}\bar{m}^* + \tilde{b}_{\Gamma_i,\perp} u^* + v\tilde{c}_{\Gamma_i,\perp}^*) = 1, \end{aligned}$$

where $\bar{h}\bar{m}^*$ denotes a particular solution to the equation $y_i = \tilde{b}_{\Gamma_i}^* h m^* \tilde{c}_{\Gamma_i}$.

A block Kaczmarz method for blind deconvolution

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Riemannian optimization formulation

$$\min \|\tilde{X} - \tilde{h}_l \tilde{m}_l^*\|_F^2, \text{ subject to } \tilde{X} \in \mathcal{N}_{\Gamma_i},$$

where $\mathcal{N}_{\Gamma_i} = \{\tilde{X} \in \mathbb{C}^{(1+\frac{l}{N_b}) \times (1+\frac{l}{N_b})} \mid \text{rank}(\tilde{X}) = 1, y_{\Gamma_i} = \tilde{b}_{\Gamma_i}^* \tilde{X} \tilde{c}_{\Gamma_i}\}$.

A block Kaczmarz method for blind deconvolution

Computation of the projection

Riemannian optimization formulation

$$\min \|\tilde{X} - \tilde{h}_l \tilde{m}_l^*\|_F^2, \text{ subject to } \tilde{X} \in \mathcal{N}_{\Gamma_i},$$

where $\mathcal{N}_{\Gamma_i} = \{\tilde{X} \in \mathbb{C}^{(1+\frac{L}{N_b}) \times (1+\frac{L}{N_b})} \mid \text{rank}(\tilde{X}) = 1, y_{\Gamma_i} = \tilde{b}_{\Gamma_i}^* \tilde{X} \tilde{c}_{\Gamma_i}\}$.

Manifold of \mathcal{N}_{Γ_i} :

- $T_{\tilde{X}} \mathcal{N}_{\Gamma_i} = \{\sigma \tilde{h} \tilde{m}^* + h_p \tilde{m}^* + \tilde{h} m_p^*\} \cap \{\tilde{X} : 0 = \tilde{b}_{\Gamma_i}^* \tilde{X} \tilde{c}_{\Gamma_i}\}$
 $= \{[\tilde{h} \quad \tilde{h}_{\perp}] \begin{bmatrix} \sigma & \alpha \\ \beta & 0 \end{bmatrix} \begin{bmatrix} \tilde{m}^* \\ \tilde{m}_{\perp}^* \end{bmatrix}\} \cap \{\tilde{X} : 0 = \tilde{b}_{\Gamma_i}^* \tilde{X} \tilde{c}_{\Gamma_i}\};$
- $N_{\tilde{X}} \mathcal{N}_{\Gamma_i} = \text{span}\{\tilde{h}_{\perp} \tilde{m}_{\perp}^*, \tilde{b}_{\Gamma_i}(:, 1) \tilde{c}_{\Gamma_i}(:, 1)^*, \dots, \tilde{b}_{\Gamma_i}(:, \frac{L}{N_b}) \tilde{c}_{\Gamma_i}(:, \frac{L}{N_b})^*\};$
- $\text{Proj}_{\tilde{X}}(G) = (I - QQ^*)G$, where Q is an orthonormal basis of $N_{\tilde{X}} \mathcal{N}_{\Gamma_i}$;
- $\text{grad } f(\tilde{X}) = \text{Proj}_{\tilde{X}}(\nabla f(\tilde{X}))$, where $\nabla f(\tilde{X}) = 2(\tilde{X} - \tilde{h}_l \tilde{m}_l^*)$;

A block Kaczmarz method for blind deconvolution

Computation of the projection

Riemannian optimization formulation

$$\min \|\tilde{X} - \tilde{h}_l \tilde{m}_l^*\|_F^2, \text{ subject to } \tilde{X} \in \mathcal{N}_{\Gamma_i},$$

where $\mathcal{N}_{\Gamma_i} = \{\tilde{X} \in \mathbb{C}^{(1+\frac{L}{N_b}) \times (1+\frac{L}{N_b})} \mid \text{rank}(\tilde{X}) = 1, y_{\Gamma_i} = \tilde{b}_{\Gamma_i}^* \tilde{X} \tilde{c}_{\Gamma_i}\}$.

Manifold of \mathcal{N}_{Γ_i} :

- $R(W)$

- Calculate $q_0 r^* = \pi_1(W)$ and $QR = \tilde{b}_{\Gamma_i}$;

- $\alpha_1 = y_{\Gamma_i,1} / (\tilde{b}_{\Gamma_i}(:,1)^* q_0 r^* \tilde{c}_{\Gamma_i}(:,1))$;

- **for** $k = 2, \dots, \frac{L}{N_b}$

$$\alpha_k = \frac{(y_{\Gamma_i,k} - \sum_{j=0}^{k-2} \alpha_{j+1} \tilde{b}_{\Gamma_i}(:,k) q_j r^* \tilde{c}_{\Gamma_i}(:,k))}{\tilde{b}_{\Gamma_i}(:,k)^* q_{k-1} r^* \tilde{c}_{\Gamma_i}(:,k)};$$

end for

- $R(W) = (\sum_{k=1}^{\frac{L}{N_b}} \alpha_k q_{k-1}) r^*$;

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