Solving PhaseLift by low-rank Riemannian optimization methods

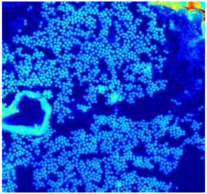
Wen Huang[*], Kyle A. Gallivan[♭] and Xiangxiong Zhang[†]

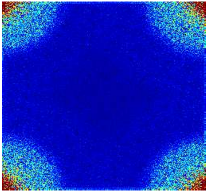
[*]Université catholique de Louvain [♭]Florida State University [†] Purdue University

June 2016

Introduction

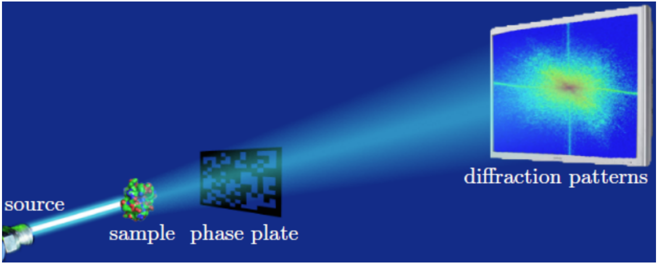
- The Phase Retrieval problem concerns recovering a signal given the modulus of its linear transform, e.g., Fourier or wavelet transform;
- Applications:
  - X-ray crystallography imaging [Har93];
  - Diffraction imaging [BDP$^+$07];
  - Optics [Wal63];
  - Microscopy [MISE08].





Problem formulation

For a signal x of size n, only $b_i = |a_i^* x|^2, i = 1, \ldots, m$ are observed, e.g., X-ray diffraction image.



Candés et al.

- Find $x \in \mathbb{C}^n$ such that $|a_i^* x|^2 = b_i, i = 1, \ldots, m$;
- $|a_i^* x|^2 = a_i^* x x^* a_i$;
- Let $A = \begin{bmatrix} a_1 & a_2 & \ldots & a_m \end{bmatrix}^T$ and $b = \begin{bmatrix} b_1 & b_2 & \ldots & b_m \end{bmatrix}^T$;
- Find $x \in \mathbb{C}^n$ such that $\mathrm{diag}(A x x^* A^*) = b$;

Problem formulation

Feasible problem

$$\text{Find } x \in \mathbb{C}^n \text{ such that } \operatorname{diag}(Axx^*A^*) = b;$$

PhaseLift framework [CESV13][1]

- Lifting (quadratic to linear): $X := xx^* \Rightarrow \operatorname{diag}(AXA^*) =: \mathcal{A}(X)$;
- Find $X \in \mathbb{C}^{n \times n}$ such that $\mathcal{A}(X) = b$, $X \succcurlyeq 0$ and $\operatorname{rank}(X) = 1$;
- Convex optimization problem: $\min_{X \in \mathbb{S}_+^n} \|b - \mathcal{A}(X)\|_2^2 + \kappa \operatorname{trace}(X)$;
- Exactly recovery: [CSV13, CL13, DH14];
- The dimension of domain;

[1] [CESV13]: E. J. Candés, Y. C. Eldar, T.Strohmer, and V. Voroninski, Phase retrieval via matrix completion, *SIAM Journal on Imaging Sciences*, 6(1):199-225, 2013

An equivalent cost function

$$\mathcal{A} : \mathbb{S}_+^n \to \mathbb{R} : X \mapsto \mathcal{A}(X) = \|b - \mathcal{A}(X)\|_2^2 + \kappa \operatorname{trace}(X).$$

- The cost function: in form of $H : \mathbb{S}_+^n \to \mathbb{R} : X \mapsto H(X)$;
- $\operatorname{rank}(X) \leq p \Longrightarrow \exists Y \in \mathbb{C}^{n \times p}$ such that $X = YY^*$;
- An alternate cost function to H can be used[2]:

$$F_p : \mathbb{C}^{n \times p} \to \mathbb{R} : Y_p \mapsto H(Y_p Y_p^*);$$

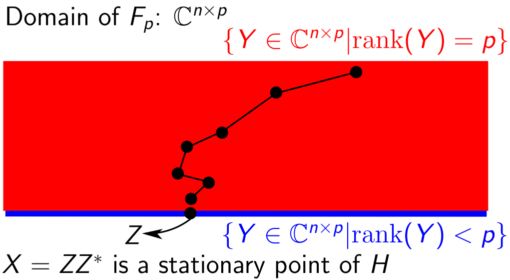
- $p = 1$ is equivalent to the cost function without lifting;
- $p > 1$ brings benefits.

[2]This idea has been used in [BM03, JBAS10] for real numbers.

Optimality condition

Theorem

Suppose Y_p is a rank deficient minimizer of F_p. Then $X = Y_p Y_p^*$ is a stationary point of H. If furthermore H is convex, then X is a global minimizer of H over $\mathbb{S}_+^n$.



A version for real numbers of this Theorem can be found at [JBAS10].

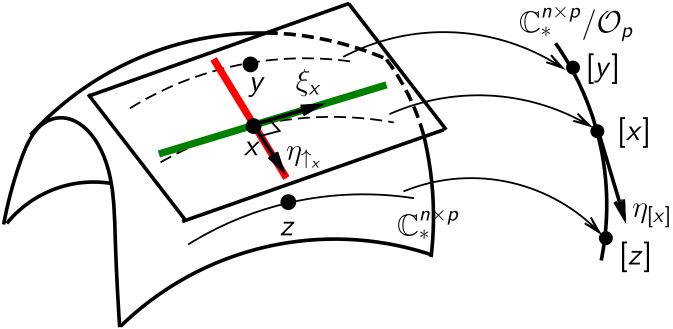
Optimization

- Equivalence: $F_p(Y_p) = F_p(Y_p O_p)$, where $O_p \in \mathcal{O}_p$ and $\mathcal{O}_p$ is the set of all p-by-p unitary matrices;
- All minimizers: degenerate;
- Optimization on a quotient manifold;
- Reduce the rank p;
- $\mathbb{C}_*^{n \times p} / \mathcal{O}_p = \{[X] \mid X \in \mathbb{C}_*^{n \times p}\}$, where $\mathbb{C}_*^{n \times p}$ is the noncompact Stiefel manifold and $[X] = \{XO \mid O \in \mathcal{O}_p\}$;

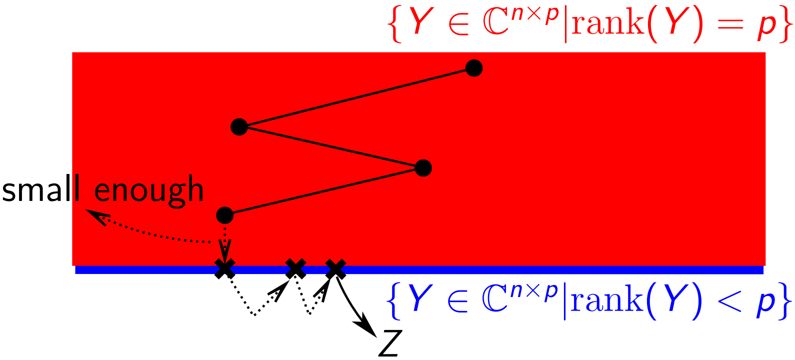
Riemannian optimization on a quotient manifold

- Work with representatives;
- Green line: vertical space;
- Red line: horizontal space;

Riemannian metric:
$\langle \eta_X, \xi_X \rangle_X = \mathrm{Real}(\mathrm{tr}((X^* X)\eta_X^* \xi_X))$;



Rank reduce method



- The desired rank is known;
- Reduce rank by eliminating small singular values;

Initial iterate

Feasible problem

Find $x \in \mathbb{C}^n$ such that $\mathrm{diag}(Axx^*A^*) = b$;

- Algorithm:

  Orthonormal matrix $Y \in \mathbb{C}^{n \times p}$.
  for i = 1, ..., N **do**
  $Y \leftarrow \mathrm{orth}(A^* \, \mathrm{Diag}(b) AY)$;
  end for
  $Y_p^{(0)} \leftarrow Y$;

- Initial iterate: an orthonormal basis of the space of the p largest eigenvectors of $A^* \mathrm{Diag}(b) A$;
- [CLS16, Algorithm 1][3] uses $p = 1$

[3][CLS16]: E. J. Candés, X. Li, and M. Soltanolkotabi, Phase retrieval via Wirtinger flow, *IEEE Transactions on Information Theory*, 64(4):1985-2007, 2016

Gap

Theorem

Suppose Y_p is a rank deficient minimizer of F_p. Then $X = Y_p Y_p^*$ is a stationary point of H. If furthermore H is convex, then X is a global minimizer of H over $\mathbb{S}_+^n$.

- It is unknown whether the proposed algorithm is guaranteed to find a rank deficient minimizer of F_p.

Experiments

- Fourier transformation;
- Initial rank $p_0 = 4$;
- Complexity per iteration: $O(pm\log(n))$, if $p \ll n$;
- Limited-memory version of Riemannian BFGS method [HGA15, HAG16]
- Stopping criterion: $\|\operatorname{grad} F(X_k)\| < 10^{-8}$;

Comparison with an existing method

Fast iterative shrinkage-thresholding algorithm [BT09]

- A state-of-the-art algorithm for convex programming;
- Too expensive: 100 by 100 image requires solving a 10000 by 10000 semidefinite programming problem;
- A low-rank version is used, denoted by LR-FISTA, used in [CESV13];
- Stopping criterion: $\|X_{k+1} - X_k\|_F / \|X_k\|_F < 10^{-6}$;

Comparison with an existing method

Table: Comparisons of Riemannian method and LR-FISTA for the noiseless PhaseLift problem with $n_1 = n_2 = 64$, $n = n_1 n_2 = 4096$ and several values of rank in LR-FISTA. RMSE denotes the relative mean-square error: $\min_{|a|=1} \|ax - x_*\|_2 / \|x_*\|_2$.

noiseless	R-method	LR-FISTA 1	2	4	8
iter	104	492	484	521	1538
nf	108	1064	1004	1113	3323
ng	104	532	502	556	1661
f_f	3.88_{-15}	3.54_{-12}	3.20_{-12}	7.41_{-12}	1.52_{-11}
RMSE	1.93_{-7}	6.20_{-6}	4.68_{-6}	1.59_{-5}	8.92_{-5}
t (sec)	8.37_{-1}	59.6	53.3	80.7	337

Performance on natural images

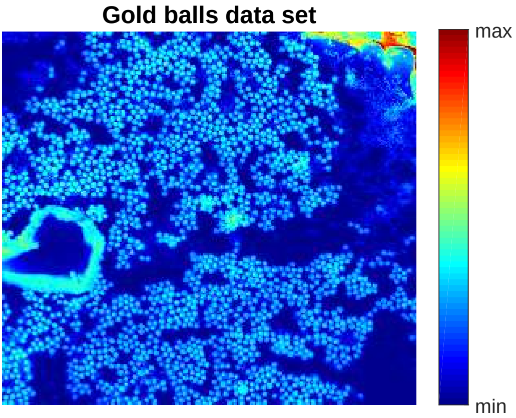


Figure: A gold balls data set image of 256 by 256 pixels. $n = 65536$. The values of pixels are complex numbers. The computational times of initial iterate and R-method are 4.86 seconds and 7.61 seconds, respectively.

Performance on natural images

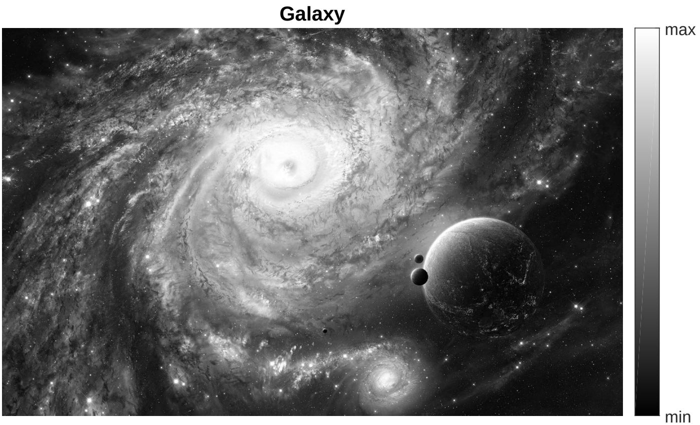


Figure: A gray galaxy image of 1800 by 2880 pixels. $n = 5184000$. The values of pixels are real numbers. The computational times of initial iterate and R-method are 536.6 seconds and 1826 seconds, respectively.

Conclusions

- Briefly introduced the phase retrieval problem and the PhaseLift framework;
- Suggested an equivalent cost function $F_p(Y_p) := H(Y_p Y_p^*)$;
- Gave an optimality condition for a stationary point of H using F_p;
- Optimized the cost function F_p by Riemannian optimization methods and a rank reduce strategy;
- Showed the efficiency and effectiveness by experiments;
- Matlab codes:
  `http://www.math.fsu.edu/~whuang2/papers/SPLRROMCSC.htm`

References I

O. Bunk, A. Diaz, F. Pfeiffer, C. David, B. Schmitt, D. K. Satapathy, and J. F. van der Veen.
Diffractive imaging for periodic samples: retrieving one-dimensional concentration profiles across microfluidic channels.
Acta crystallographica. Section A, Foundations of crystallography, 63(Pt 4):306–314, July 2007.
doi:10.1107/S0108767307021903.

S. Burer and R. D. C. Monteiro.
A nonlinear programming algorithm for solving semidefinite programs via low-rank factorization.
Mathematical Programming, 95(2):329–357, February 2003.
doi:10.1007/s10107-002-0352-8.

A. Beck and M. Teboulle.
A fast iterative shrinkage-thresholding algorithm for linear inverse problems.
SIAM Journal on Imaging Sciences, 2(1):183–202, January 2009.
doi:10.1137/080716542.

E. J. Candès, Y. C. Eldar, T. Strohmer, and V. Voroninski.
Phase retrieval via matrix completion.
SIAM Journal on Imaging Sciences, 6(1):199–225, 2013.
arXiv:1109.0573v2.

E. J. Candès and X. Li.
Solving quadratic equations via phaselift when there are about as many equations as unknowns.
Foundations of Computational Mathematics, June 2013.
doi:10.1007/s10208-013-9162-z.

E. J. Candés, X. Li, and M. Soltanolkotabi.
Phase retrieval via Wirtinger flow: theory and algorithms.
IEEE Transactions on Information Theory, 64(4):1985–2007, 2016.

References II

E. J. Candès, T. Strohmer, and V. Voroninski.
PhaseLift : Exact and stable signal recovery from magnitude measurements via convex programming.
Communications on Pure and Applied Mathematics, 66(8):1241–1274, 2013.

L. Demanet and P. Hand.
Stable optimizationless recovery from phaseless linear measurements.
Journal of Fourier Analysis and Applications, 20(1):199–221, 2014.

Wen Huang, P.-A. Absil, and K. A. Gallivan.
A Riemannian BFGS method for nonconvex optimization problems.
Lecture Notes in Computational Science and Engineering, To appear, 2016.

R. W. Harrison.
Phase problem in crystallography.
Journal of the Optical Society of America A, 10(5):1046–1055, May 1993.
doi:10.1364/JOSAA.10.001046.

Wen Huang, K. A. Gallivan, and P.-A. Absil.
A Broyden Class of Quasi-Newton Methods for Riemannian Optimization.
SIAM Journal on Optimization, 25(3):1660–1685, 2015.

M. Journée, F. Bach, P.-A. Absil, and R. Sepulchre.
Low-rank optimization on the cone of positive semidefinite matrices.
SIAM Journal on Optimization, 20(5):2327–2351, 2010.

References III

J. Miao, T. Ishikawa, Q. Shen, and T. Earnest.
Extending X-ray crystallography to allow the imaging of noncrystalline materials, cells, and single protein complexes.
Annual review of physical chemistry, 59:387–410, January 2008.
doi:10.1146/annurev.physchem.59.032607.093642.

A. Walther.
The question of phase retrieval in optics.
Optica Acta: International Journal of Optics, 10(1):41–49, January 1963.
doi:10.1080/713817747.