A limited-memory Riemannian symmetric rank-one trust-region method with a restart strategy

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Joint work with Kyle Gallivan

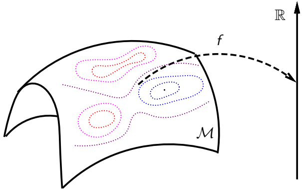
Riemannian Optimization

Problem Statement

Problem: Given $f(x) : \mathcal{M} \to \mathbb{R}$, solve

$$\min_{x \in \mathcal{M}} f(x)$$

where $\mathcal{M}$ is a Riemannian manifold.



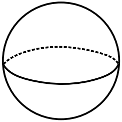
Unconstrained optimization problem on a constrained space.

Riemannian manifold = manifold + Riemannian metric

Riemannian Manifold

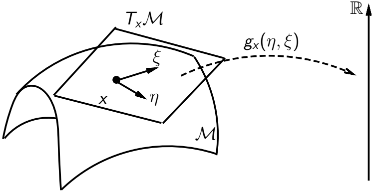
Problem Statement

Manifolds:



- Stiefel manifold: $\mathrm{St}(p, n) = \{X \in \mathbb{R}^{n \times p} | X^T X = I_p\}$;
- Grassmann manifold $\mathrm{Gr}(p, n)$: all p-dimensional subspaces of $\mathbb{R}^n$;
- And many more.

Riemannian metric:



A Riemannian metric, denoted by g, is a smoothly-varying inner product on the tangent spaces;

Riemannian Manifold

A Nonexhaustive Review

- Steepest descent: Smith 1994; Helmke-Moore 1994; Iannazzo-Porcelli 2019;
- Conjugate gradient: Smith 1994; Gallivan-Absil 2010; Ring-Wirth 2012; Sato-Iwai 2015;
- Trust region Newton: Absil-Baker-Gallivan 2007;
- Quasi-Newton:
  - BFGS Ring-Wirth 2012;
  - Trust-region SR1 Huang-Absil-Gallivan 2015;
  - Broyden family (including BFGS) Huang-Absil-Gallivan 2015;
  - BFGS, LBFGS Huang-Absil-Gallivan 2018;
  - Limited-memory trust-region SR1 Huang-Gallivan 2022;

Riemannian Manifold

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  - Limited-memory trust-region SR1 Huang-Gallivan 2022;

A Limited-memory Trust-region Symmetric Rank-One Update Method with a Restart Strategy (LRTRSR1-R)

The idea is based on

- Intrinsic representation of tangent vectors[1];
- An efficient solver for the Euclidean LSR1 subproblem[2];
- Slight modifications to LRTRSR1;

[1] W. Huang, P.-A. Absil, and K. A. Gallivan. Intrinsic representation of tangent vectors and vector transport on matrix manifolds. Numerische Mathematik, 136(2):523-543, 2017.

[2] Johannes Brust, Jennifer B. Erway, and Roummel F. Marcia. On solving L-SR1 trust- region subproblems. Computational Optimization and Applications, 66(2):245-266, Mar 2017

Euclidean TRSR1 to Riemannian TRSR1

Euclidean trust region SR1

Require: Initial iterate x_0;

1, Set $k \leftarrow 0$;

while not accurate enough **do**

2, Compute s_k by approximately solving

$$\operatorname*{argmin}_{\|s\| \leq \Delta_k} \nabla f(x_k)^T s + \frac{1}{2} s^T B_k s;$$

3, $\rho_k \leftarrow \frac{f(x_k) - f(x_k + s_k)}{m_k(0) - m_k(s_k)}$;

4, $x_{k+1} \leftarrow \begin{cases} x_k + s_k & \text{if } \rho_k > c \\ x_k & \text{otherwise.} \end{cases}$

5, Update radius to get Δ_{k+1};

6, Update B_k to get B_{k+1};

7, $k \leftarrow k + 1$;

end while

Riemannian trust region SR1

Require: Initial iterate x_0;

1, Set $k \leftarrow 0$;

while not accurate enough **do**

2, Compute $\mathfrak{s}_k$ by approximately solving

$$\operatorname*{argmin}_{\|\mathfrak{s}\| \leq \Delta_k, \mathfrak{s} \in \mathrm{T}_{x_k} \mathcal{M}} g(\operatorname{grad} f(x_k), \mathfrak{s}) + \frac{1}{2} g(\mathfrak{s}, \mathcal{B}_k \mathfrak{s});$$

3, $\rho_k \leftarrow \frac{f(x_k) - f(R_{x_k}(\mathfrak{s}_k))}{m_k(0) - m_k(\mathfrak{s}_k)}$;

4, $x_{k+1} \leftarrow \begin{cases} R_{x_k}(\mathfrak{s}_k) & \text{if } \rho_k > c \\ x_k & \text{otherwise.} \end{cases}$

5, Update radius to get Δ_{k+1};

6, Update $\mathcal{B}_k$ to get $\mathcal{B}_{k+1}$;

7, $k \leftarrow k + 1$;

end while

Trust-region LSR Subproblem

Euclidean Version

Local problem in Euclidean trust region:

$$\min_{\|s\| \leq \Delta} m(s) = s^T g + \frac{1}{2} s^T B s;$$

- at k-th iteration: $\Delta = \Delta_k$, $g = \nabla f(x_k)$ and $B = B_k$;
- SR1 update: $(y_k = \nabla f(x_{k+1}) - \nabla f(x_k))$

$$B_{k+1} = B_k + \frac{(y_k - B_k s_k)(y_k - B_k s_k)^T}{(y_k - B_k s_k)^T s_k};$$

- Compact representation:

$$\begin{aligned} B_{k+1} &= B_{k-m+1} + (Y_{k+1} - B_{k-m+1} S_{k+1}) \\ &\quad (D_{k+1} + L_{k+1} + L_{k+1}^T - S_{k+1}^T B_{k-m+1} S_{k+1})^{-1} (Y_{k+1} - B_{k-m+1} S_{k+1})^T; \\ &= \gamma I + \Phi M^{-1} \Phi^T; \end{aligned}$$

where $Y_{k+1} = [y_{k-m+1}, \dots, y_k]$, $S_{k+1} = [s_{k-m+1}, \dots, s_k]$, and $\gamma I = B_{k-m+1}$.

Trust-region LSR Subproblem

Euclidean Version

Local problem in Euclidean trust region:

$$\min_{\|s\| \leq \Delta} m(s) = s^T g + \frac{1}{2} s^T B s;$$

Theorem ([NW06])

The vector s^* is a global solution of the trust region subproblem

$$\min_{\|s\| \leq \Delta} \nabla f(x)^T s + \frac{1}{2} s^T B s$$

if and only if s^* is feasible and there is a scalar $\lambda \geq 0$ such that the following conditions hold:

$$(B + \lambda I) s^* = -\nabla f(x), \quad \lambda(\Delta - \|s^*\|) = 0, \quad (B + \lambda I) \succeq 0.$$

Trust-region LSR Subproblem

Euclidean Version

Local problem in Euclidean trust region:

$$\min_{\|s\| \leq \Delta} m(s) = s^T g + \frac{1}{2} s^T B s;$$

Eigenvalue decomposition

$$B = \gamma I + \Phi M^{-1} \Phi^T = \begin{bmatrix} P & P_\perp \end{bmatrix} \left(\gamma I + \begin{bmatrix} S & \\ & 0_{(n-m)\times(n-m)} \end{bmatrix} \right) \begin{bmatrix} P & P_\perp \end{bmatrix}^T,$$

where $\Phi = QR$ is qr decomposition, $RM^{-1}R^T = USU^T$ is eigenvalue decomposition, $P = QU$ and $[P, P_\perp]$ is orthonormal.

- Eigenvalues of B: $\Lambda = \gamma + \mathrm{diag}(s_1, \dots, s_m, 0, \dots, 0)$;
- $s(\lambda) = -(B + \lambda I)^{-1} g$ for finding the solution s^*;
- $h(\lambda) = \frac{1}{\|s(\lambda)\|} - \frac{1}{\Delta}$ for finding the appropriate λ;
- Complexity: $O(nm) + O(m^3)$;

Trust-region LSR Subproblem

Riemannian Version

Local problem in Riemannian trust region:

$$\min_{\|\mathfrak{s}\| \leq \Delta, \mathfrak{s} \in \mathrm{T}_x \mathcal{M}} m(\mathfrak{s}) = g_x(\mathfrak{s}, g) + \frac{1}{2} g_x(\mathfrak{s}, \mathcal{B}\mathfrak{s});$$

Turn Riemannian LSR1 subproblem into Euclidean version:

- Q_x: an orthonormal basis of $\mathrm{T}_x \mathcal{M}$;
- $\mathrm{T}_x \mathcal{M} = \{Q_x c : c \in \mathbb{R}^d\}$;
- Replace $\mathfrak{s}$ by $Q_x c$:

$$\min_{\|Q_x c\| \leq \Delta} c^T Q_x^\flat g + \frac{1}{2} c^T (Q_x^\flat \mathcal{B} Q) c = \min_{\|c\| \leq \Delta} c^T \hat{g} + \frac{1}{2} c^T \hat{\mathcal{B}} c;$$

- Solved by Euclidean version algorithm

Trust-region LSR Subproblem

Riemannian Version

Local problem in Riemannian trust region:

$$\min_{\|\mathfrak{s}\| \leq \Delta, \mathfrak{s} \in \mathrm{T}_x \mathcal{M}} m(\mathfrak{s}) = g_x(\mathfrak{s}, g) + \frac{1}{2} g_x(\mathfrak{s}, \mathcal{B}\mathfrak{s});$$

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- Solved by Euclidean version algorithm

Orthonormal basis Q_x is crucial!

Intrinsic Representation of Tangent Vector

- Given $\eta_x \in \mathrm{T}_x \mathcal{M}$, compute $c_x = Q_x^\flat \eta_x$, i.e., find $c_x \in \mathbb{R}^d$ such that $\eta_x = Q_x c_x$;
- Given $c_x \in \mathbb{R}^d$, compute $\eta_x = Q_x c_x$;

- Stiefel manifold as an example: $\mathrm{St}(p, n) = \{X \in \mathbb{R}^{n \times p} : X^T X = I_p\}$.
- Tangent space at $X \in \mathrm{T}_X \mathcal{M}$ is $\mathrm{T}_X \mathrm{St}(p, n) = \{X\Omega + X_\perp K \mid \Omega^T = -\Omega, X^T X_\perp = 0\}$;

$$Q_x = \left\{ \begin{bmatrix} X & X_\perp \end{bmatrix} \left[\begin{array}{cccc} 0 & 1 & \dots & 0 \\ -1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \\ \hline 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{array}\right], \dots, \begin{bmatrix} X & X_\perp \end{bmatrix} \left[\begin{array}{cccc} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \\ \hline 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{array}\right] \right\}$$

Intrinsic Representation of Tangent Vector

$$\mathrm{T}_X \mathrm{St}(p,n) = \{X\Omega + X_\perp K \mid \Omega^T = -\Omega, X^T X_\perp = 0\};$$

Extrinsic η_X:

$$\eta_X = \begin{bmatrix} X & X_\perp \end{bmatrix} \begin{bmatrix} \Omega \\ K \end{bmatrix}$$

$$= \begin{bmatrix} X & X_\perp \end{bmatrix} \begin{bmatrix} 0 & a_{12} & \dots & a_{1p} \\ -a_{12} & 0 & \dots & a_{2p} \\ \dots & \dots & \dots & \dots \\ -a_{1p} & -a_{2p} & \dots & 0 \\ b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \dots & \dots & \dots & \dots \\ b_{(n-p)1} & b_{(n-p)2} & \dots & b_{(n-p)p} \end{bmatrix}$$

Intrinsic c_X:

$$c_X = \begin{bmatrix} a_{12} \\ a_{13} \\ a_{23} \\ \vdots \\ a_{(p-1)p} \\ b_{11} \\ b_{21} \\ \vdots \\ b_{(n-p)p} \end{bmatrix}$$

Intrinsic Representation of Tangent Vector

Question

Extrinsic representation $\eta_X \Longleftrightarrow$ Intrinsic representation c_X

- $\eta_X = \begin{bmatrix} X & X_\perp \end{bmatrix} \begin{bmatrix} \Omega \\ K \end{bmatrix} \Leftrightarrow \begin{bmatrix} \Omega \\ K \end{bmatrix} \Leftrightarrow c_X$

Intrinsic Representation of Tangent Vector

Question

Extrinsic representation $\eta_X \Longleftrightarrow$ Intrinsic representation c_X

- $\eta_X = \begin{bmatrix} X & X_\perp \end{bmatrix} \begin{bmatrix} \Omega \\ K \end{bmatrix} \Leftrightarrow \begin{bmatrix} \Omega \\ K \end{bmatrix} \Leftrightarrow c_X$

- Apply Householder transformation to X, (Done in retraction)

$$Q_p^T Q_{p-1}^T \dots Q_1^T X = R = I_{n \times p}.$$

Intrinsic Representation of Tangent Vector

Question

Extrinsic representation $\eta_X \Longleftrightarrow$ Intrinsic representation c_X

- $\eta_X = \begin{bmatrix} X & X_\perp \end{bmatrix} \begin{bmatrix} \Omega \\ K \end{bmatrix} \Leftrightarrow \begin{bmatrix} \Omega \\ K \end{bmatrix} \Leftrightarrow c_X$

- Apply Householder transformation to X, (Done in retraction)

$$Q_p^T Q_{p-1}^T \dots Q_1^T X = R = I_{n \times p}.$$

- $\begin{bmatrix} X & X_\perp \end{bmatrix} = Q_1 Q_2 \dots Q_p$ (Do not compute)

Intrinsic Representation of Tangent Vector

Question

Extrinsic representation $\eta_X \Longleftrightarrow$ Intrinsic representation c_X

- $\eta_X = \begin{bmatrix} X & X_\perp \end{bmatrix} \begin{bmatrix} \Omega \\ K \end{bmatrix} \Leftrightarrow \begin{bmatrix} \Omega \\ K \end{bmatrix} \Leftrightarrow c_X$
- Apply Householder transformation to X, (Done in retraction)

$$Q_p^T Q_{p-1}^T \dots Q_1^T X = R = I_{n \times p}.$$

- $\begin{bmatrix} X & X_\perp \end{bmatrix} = Q_1 Q_2 \dots Q_p$ (Do not compute)
- Extrinsic to Intrinsic: $Q_p^T Q_{p-1}^T \dots Q_1^T \eta_X = \begin{bmatrix} \Omega \\ K \end{bmatrix}$ and reshape to c_X; $(4np^2 - 2p^3)$ flops
- Intrinsic to Extrinsic: reshape c_X and $\eta_X = Q_1 Q_2 \dots Q_p \begin{bmatrix} \Omega \\ K \end{bmatrix}$; $(4np^2 - 2p^3)$ flops

Intrinsic Representation of Tangent Vector

- The basis Q_x is orthonormal with respect to the canonical metric:

$$g_X(\eta_X, \xi_X) = \mathrm{trace}\left(\eta_X^T \left(I_n - \frac{1}{2} X X^T\right) \xi_X\right), \forall \eta_X, \xi_X \in \mathrm{T}_X \, \mathrm{St}(p, n).$$

- The above idea can be applied to
  - Fixed-rank real matrices $\mathbb{R}_r^{m \times n} = \{X \in \mathbb{R}^{m \times n} : \mathrm{rank}(X) = r\}$;
  - Fixed-rank symmetric positive semidefinite matrices: $\mathcal{S}_r^n = \{X \in \mathbb{R}^{n \times n} : X = X^T, X \succeq 0, \mathrm{rank}(X) = r\}$;
  - Fixed-rank complex matrices;
  - Fixed-rank Hermitian positive semidefinite matrices;

Riemannian LSR1 Update

$$\min_{\|Q_x c\| \leq \Delta} c^T Q_x^\flat g + \frac{1}{2} c^T (Q_x^\flat \mathcal{B} Q_x) c = \min_{\|c\| \leq \Delta} c^T \hat{g} + \frac{1}{2} c^T \hat{\mathcal{B}} c;$$

$$\begin{aligned}
\tilde{\mathcal{B}}_k =& \mathcal{B}_{k-1} + \frac{(\mathfrak{y}_{k-1} - \mathcal{B}_{k-1}\mathfrak{s}_{k-1})(\mathfrak{y}_{k-1} - \mathcal{B}_{k-1}\mathfrak{s}_{k-1})^\flat}{g(\mathfrak{s}_{k-1}, \mathfrak{y}_{k-1} - \mathcal{B}_{k-1}\mathfrak{s}_{k-1})}, \\
\mathcal{B}_k =& \mathcal{T}_{\mathrm{S}_{\mathfrak{s}_{k-1}}} \circ \tilde{\mathcal{B}}_k \circ \mathcal{T}_{\mathrm{S}_{\mathfrak{s}_{k-1}}}^{-1},
\end{aligned}$$

where $\mathfrak{y}_{k-1} = \mathcal{T}_{\mathrm{S}_{\mathfrak{s}_{k-1}}}^{-1} \operatorname{grad} f(R_{x_{k-1}}(\mathfrak{s}_{k-1})) - \operatorname{grad} f(x_{k-1}) \in \mathrm{T}_{x_{k-1}} \mathcal{M}$, $\mathfrak{s}_{k-1} \in \mathrm{T}_{x_{k-1}} \mathcal{M}$, $\mathcal{B}_k$ is a linear operator on $\mathrm{T}_{x_k} \mathcal{M}$ and $\mathcal{T}_{\mathrm{S}}$ is an isometric vector transport.

Riemannian LSR1 Update

Compact representation

Theorem

Let $\mathcal{B}_{k-\ell}$ be a symmetric linear operator on $\mathrm{T}_{x_{k-\ell}}\mathcal{M}$ and $\mathcal{T}_{\mathrm{S}}$ is an isometric vector transport. Suppose $g(\mathfrak{s}_{k-i}^{(k)}, \mathfrak{y}_{k-i}^{(k)} - \mathcal{B}_{k-i}^{(k)}\mathfrak{s}_{k-i}^{(k)}) \neq 0$ for $i = 1, \dots, \ell$. Then the linear operator $\mathcal{B}_k^{(k)}$ is given by

$$\mathcal{B}_k^{(k)} = \mathcal{B}_{k-\ell}^{(k)} + \left(\mathcal{Y}_\ell^{(k)} - \mathcal{B}_{k-\ell}^{(k)}\mathcal{S}_\ell^{(k)}\right)\left(M_\ell^{(k)}\right)^{-1}\left(\mathcal{Y}_\ell^{(k)} - \mathcal{B}_{k-\ell}^{(k)}\mathcal{S}_\ell^{(k)}\right)^\flat,$$

where $D_\ell^{(k)} = \mathrm{diag}((\mathfrak{s}_{k-\ell}^{(k)})^\flat\mathfrak{y}_{k-\ell}^{(k)} \dots, (\mathfrak{s}_{k-1}^{(k)})^\flat\mathfrak{y}_{k-1}^{(k)})$, $\mathcal{S}_\ell^{(k)} = [\mathfrak{s}_{k-\ell}^{(k)}, \dots, \mathfrak{s}_{k-1}^{(k)}]$, $\mathcal{Y}_\ell^{(k)} = [\mathfrak{y}_{k-\ell}^{(k)}, \dots, \mathfrak{y}_{k-1}^{(k)}]$, $(L_\ell)_{i,j}^{(k)} = (\mathfrak{s}_{k-\ell-1+i}^{(k)})^\flat\mathfrak{y}_{k-\ell-1+j}^{(k)}$ if $i > j$, $(L_\ell)_{i,j}^{(k)} = 0$ otherwise, and $M_\ell^{(k)} = D_\ell^{(k)} + L_\ell^{(k)} + (L_\ell^{(k)})^T - (\mathcal{S}_\ell^{(k)})^\flat\mathcal{B}_{k-\ell}^{(k)}\mathcal{S}_\ell^{(k)}$ is nonsingular.

- $\mathcal{B}_{k-i}^{(k)}$ denotes $\mathcal{T}_{\mathrm{S}_{\mathfrak{s}_{k-1}}} \circ \dots \circ \mathcal{T}_{\mathrm{S}_{\mathfrak{s}_{k-i}}} \circ \mathcal{B}_{k-i} \circ \mathcal{T}_{\mathrm{S}_{\mathfrak{s}_{k-i}}}^{-1} \circ \dots \circ \mathcal{T}_{\mathrm{S}_{\mathfrak{s}_{k-1}}}^{-1}$;
- $\mathfrak{y}_{k-i}^{(k)}$ denotes $\mathcal{T}_{\mathrm{S}_{\mathfrak{s}_{k-1}}} \circ \dots \circ \mathcal{T}_{\mathrm{S}_{\mathfrak{s}_{k-i}}}\mathfrak{y}_{k-i}$;
- $\mathfrak{s}_{k-i}^{(k)}$ denotes $\mathcal{T}_{\mathrm{S}_{\mathfrak{s}_{k-1}}} \circ \dots \circ \mathcal{T}_{\mathrm{S}_{\mathfrak{s}_{k-i}}}\mathfrak{s}_{k-i}$.

Riemannian LSR1 Update

Compact representation

Vector transport by parallelization:

$$\mathcal{T}_{\mathrm{S}_{\eta_x}} \xi_x = Q_y Q_x^{\flat} \xi_x$$

where $y = R_x(\eta_x)$;

- We have

$$\begin{aligned}
\mathcal{B}_{k-i}^{(k)} =& \mathcal{T}_{\mathrm{S}_{\mathfrak{s}_{k-1}}} \circ \dots \circ \mathcal{T}_{\mathrm{S}_{\mathfrak{s}_{k-i}}} \circ \mathcal{B}_{k-i} \circ \mathcal{T}_{\mathrm{S}_{\mathfrak{s}_{k-i}}}^{-1} \circ \dots \circ \mathcal{T}_{\mathrm{S}_{\mathfrak{s}_{k-1}}}^{-1} \\
=& Q_{x_k} Q_{x_{k-1}}^{\flat} \dots Q_{x_{k-i+1}} Q_{x_{k-i}}^{\flat} \mathcal{B}_{k-i} Q_{x_{k-i}} Q_{x_{k-i+1}}^{\flat} \dots Q_{x_{k-1}} Q_{x_k}^{\flat} \\
=& Q_{x_k} Q_{x_{k-i}}^{\flat} \mathcal{B}_{k-i} Q_{x_{k-i}} Q_{x_k}^{\flat};
\end{aligned}$$

- Similarly, $\mathfrak{y}_{k-i}^{(k)} = Q_{x_k} Q_{x_{k-i}}^{\flat} \mathfrak{y}_{k-i}$ and $\mathfrak{s}_{k-i}^{(k)} = Q_{x_k} Q_{x_{k-i}}^{\flat} \mathfrak{s}_{k-i}$.

Riemannian LSR1 Update

Compact representation

Compact representation:

$$\mathcal{B}_{k+1} = Q_{x_{k+1}} \left(\gamma_{k+1} I + \Phi_k (M_m^{(k+1)})^{-1} \Phi_k^T \right) Q_{x_{k+1}}^\flat$$

where $\Phi_k = [Q^\flat_{x_{k-m+1}} (\mathfrak{y}_{k-m+1} - \gamma_{k+1} \mathfrak{s}_{k-m+1}), \ldots, Q^\flat_{x_k} (\mathfrak{y}_k - \gamma_{k+1} \mathfrak{s}_k)]$.

$$\min_{\|Q_x c\| \leq \Delta} c^T Q_x^\flat g + \frac{1}{2} c^T (Q_x^\flat \mathcal{B} Q_x) c = \min_{\|c\| \leq \Delta} c^T \hat{g} + \frac{1}{2} c^T \left(\gamma I + \Phi M^{-1} \Phi^T \right) c$$

- $s_k = Q^\flat_{x_k} \mathfrak{s}_k$;
- $g_k = Q^\flat_{x_k} \operatorname{grad} f(x_k)$;
- $y_k = g_{k+1} - g_k$;
- Compute B_k as the Euclidean version using s_k, y_k;

Riemannian LSR1 Update

Initial Hessian Approximation γ

Set

$$\gamma = \operatorname{sign}(g(\mathfrak{y}, \mathfrak{s})) \min \left(\tau_\gamma \frac{|g(\mathfrak{y}, \mathfrak{s})|}{g(\mathfrak{s}, \mathfrak{s})}, \frac{g(\mathfrak{y}, \mathfrak{y})}{|g(\mathfrak{y}, \mathfrak{s})|} \right),$$

where τ_γ is a prescribed parameter;

- LBFGS: Initial inverse Hessian approximation is $\frac{g(\mathfrak{s},\mathfrak{y})}{g(\mathfrak{y},\mathfrak{y})}$;
- LSR1: Initial Hessian approximation is $\frac{g(\mathfrak{y},\mathfrak{y})}{g(\mathfrak{s},\mathfrak{y})}$;
- If $\|\mathfrak{y}\| \leq L\|\mathfrak{s}\|$, then
  - $\left|\frac{g(\mathfrak{y},\mathfrak{s})}{g(\mathfrak{s},\mathfrak{s})}\right|$ bounded above;
  - $\left|\frac{g(\mathfrak{y},\mathfrak{y})}{g(\mathfrak{s},\mathfrak{y})}\right|$ may not bounded above;
- Bounded above: for theoretical analysis

LRTRSR1-R (Part I)

Input: Riemannian manifold $\mathcal{M}$ with Riemannian metric g; retraction R; isometric vector transport by parallelization $\mathcal{T}_S$; smooth function f on $\mathcal{M}$; initial iterate $x_0 \in \mathcal{M}$;

Choose an integer $m > 0$ and real numbers $\bar{\Delta} > 0$, $\Delta_0 \in (0, \bar{\Delta})$, $\nu \in (0,1)$, $c \in (0, 0.25)$, $\tau_1 \in (0,1)$, and $\tau_2 > 1$;

Compute $\mathrm{grad}^{\mathrm{d}} f(x_0) = Q_{x_0}^{\flat} \,\mathrm{grad}\, f(x_0) \in \mathbb{R}^d$; Set $k \leftarrow 0$, $\ell \leftarrow 0$, $\gamma_0 \leftarrow 1$;

Obtain $s_k \in \mathbb{R}^d$ by (approximately) solving

$$s_k \approx \mathop{\arg\min}_{\|s\|_2 \leq \Delta_k} f(x_k) + s^T \mathrm{grad}^{\mathrm{d}} f(x_k) + \frac{1}{2} s^T B_k s.$$

Compute $\mathfrak{s}_k = Q_{x_k} s_k$;

Set $\rho_k \leftarrow \frac{f(x_k) - f(R_{x_k}(\mathfrak{s}_k))}{m_k(0) - m_k(\mathfrak{s}_k)}$;

Set $y_k \leftarrow Q_{R_{x_k}(\mathfrak{s}_k)}^{\flat} \,\mathrm{grad}\, f(R_{x_k}(\mathfrak{s}_k)) - \mathrm{grad}^{\mathrm{d}} f(x_k)$;

if $|s_k^T (y_k - B_k s_k)| \geq \nu \|s_k\|_2 \|y_k - B_k s_k\|_2$ **then**

  if $\ell = m$ **then**

    $\gamma_{k+1} \leftarrow \mathrm{sign}(y_k^T s_k \min\left(\tau_\gamma \frac{|y_k^T s_k|}{s_k^T s_k}, \frac{y_k^T y_k}{|y_k^T s_k|}\right)$ if $|y_k^T s_k| \leq \epsilon$, and $\gamma_{k+1} \leftarrow 0$ otherwise, where ϵ denotes the machine epsilon; Set $\Psi_{k,\ell}$ and $M_{k,\ell}$ to be empty matrices;

  else

    $\gamma_{k+1} \leftarrow \gamma_k$;

  end if

  Set $\tilde{\Psi}_{k+1,\ell+1} = [\tilde{\Psi}_{k,\ell}, y_k - \gamma_{k+1} s_k]$; Computing $M_{k+1,\ell+1}$ by updating $M_{k,\ell}$; Set $\ell \leftarrow \ell + 1$;

LRTRSR1-R (Part II)

14: **else**
15: Set $\gamma_{k+1} \leftarrow \gamma_k$, $M_{k+1,\ell} \leftarrow M_{k,\ell}$;
16: **end if**
17: **if** $\rho_k > c$ **then**
18: $x_{k+1} \leftarrow R_{x_k}(\mathfrak{s}_k)$;
19: **else**
20: $x_{k+1} \leftarrow x_k$;
21: **end if**
22: **if** $\rho_k > \frac{3}{4}$ **then**
23: If $\|\eta_k\| \geq 0.8\Delta_k$, then $\Delta_{k+1} \leftarrow \min(\tau_2 \Delta_k, \overline{\Delta})$; otherwise $\Delta_{k+1} \leftarrow \Delta_k$;
24: **else if** $\rho_k < 0.1$ **then**
25: $\Delta_{k+1} \leftarrow \tau_1 \Delta_k$;
26: **else**
27: $\Delta_{k+1} \leftarrow \Delta_k$;
28: **end if**
29: $k \leftarrow k+1$, goto 2 until convergence.

Complexity:

$$f_f + g_f + r_f + \theta_f + \vartheta_f + 10md + O(m^3),$$

where f_f, g_f, r_f, θ_f, ϑ_f respectively denote the computational cost of one function evaluation, one gradient evaluation, one retraction evaluation, one $Q_x c$ evaluation and one $Q_x^\flat \eta_x$ evaluation.

Theoretical Results

Assumption

There exists a positive constant δ_r, such that the function f is Lipschitz continuously differentiable with respect to the vector transport $\mathcal{T}_{\mathrm{S}}$ in Ω_{δ_r}, where the Lipschitz continuous differentiability is defined by

$$\|\mathcal{T}_{\xi_x}^{-1}\eta_y - \eta_x\|_x \leq L_v\|\xi_x\|_x,$$

and $\Omega_{\delta_r} = \{R_x(\eta_x) : x \in \Omega, \|\eta_x\| \leq \delta_r\}$.

Theorem

Suppose above Assumption holds and the parameter $\overline{\Delta}$ in LRTRSR1-R satisfies $\overline{\Delta} \leq \delta_r$. Then there exists a constant β such that the sequence $\{\mathcal{B}_k\}$ generated by LRTRSR1-R satisfies that $\|\mathcal{B}_k\| \leq \beta$ for all k.

Theoretical Results

Lemma

Suppose that $\mathcal{T}_{\mathrm{S}} \in C^1$, $f \in C^2$ *in* $\mathcal{U}$, *and* $\mathcal{U}$ *is compact. Further suppose that there exists a positive constant* M_r *such that for any* $x \in \mathcal{U}$, *it holds that* $\{R_x(\eta_x) : \|\eta_x\| > M_r\} \cap \mathcal{U} = \emptyset$ *or* $\{R_x(\eta_x) : \|\eta_x\| \leq M_r\} \supseteq \mathcal{U}$. *Then there exists a constant* L_v *such that the inequality*

$$\|\mathcal{T}_{\eta_x}^{-1} \operatorname{grad} f(y) - \operatorname{grad} f(x)\|_x \leq L_v \|\eta_x\|_x,$$

holds for all $x \in \mathcal{U}$, $\eta_x \in \mathrm{T}_x \mathcal{M}$ *satisfying* $R_x(t\eta_x) \in \mathcal{U}$ *for all* $t \in [0, 1]$, *where* $y = R_x(\eta_x)$.

Theoretical Results

The global convergence follows from [HAG15, Theorem 1].

Corollary

Under reasonable assumptions and the parameter $\overline{\Delta}$ in LRTRSR1-R satisfies $\overline{\Delta} \leq \delta_r$. Then (i) if $f \in C^2$ is bounded below on the sublevel set Ω, then $\lim_{k\to\infty} \operatorname{grad} f(x_k) = 0$; (ii) if $f \in C^2$, $\mathcal{M}$ is compact, then $\lim_{k\to\infty} \operatorname{grad} f(x_k) = 0$, $\{x_k\}$ has at least one limit point, and every limit point of $\{x_k\}$ is a stationary point of f; and (iii) if $f \in C^2$, the sublevel set Ω is compact, f has a unique stationary point x^* in Ω, then $\{x_k\}$ converges to x^*.

Numerical Experiments

- Convex quadratic problem
- Matrix completion
- Joint diagonalization
- Phase retrieval

Numerical Experiments

Convex Quadratic Problem

Problem:

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} x^T A x,$$

where $A = A^T$ and $A \succ 0$;

- Compare with LRBFGS, LRBFGS-R, LRTRSR1, LRTRSR1-R;
- Two types of matrices A;
    - Type 1: $A \in \mathbb{R}^{200 \times 200}$, 4 different eigenvalues, condition number $\approx$ 100;
    - Type 2: $A \in \mathbb{R}^{1000 \times 1000}$, random matrices, condition number $\approx 10^6$;

Numerical Experiments

Convex Quadratic Problem

Table: An average result of 20 random runs for the convex quadratic problem with the two types of matrices A.

		LRBFGS		LRBFGS-R		LRTRSR1		LRTRSR1-R	
	m	2	4	2	4	2	4	2	4
Type 1	*iter*	27	23	30	24	37	5	37	5
	nf	31	26	34	28	38	6	38	6
	ng	28	24	31	25	38	6	38	6
	$\frac{\|\mathrm{gf}_f\|}{\|\mathrm{gf}_0\|}$	5.76_{-7}	5.19_{-7}	6.01_{-7}	5.26_{-7}	5.05_{-7}	3.25_{-9}	4.67_{-7}	2.83_{-9}
	f	6.97_{-10}	2.26_{-9}	4.07_{-9}	1.84_{-9}	4.59_{-9}	7.96_{-12}	4.08_{-9}	4.34_{-12}
Type 2	*iter*	830	733	837	815	1323	1475	747	680
	nf	887	772	920	883	1324	1476	748	681
	ng	831	734	838	816	1324	1476	748	681
	$\frac{\|\mathrm{gf}_f\|}{\|\mathrm{gf}_0\|}$	9.25_{-7}	9.18_{-7}	9.09_{-7}	9.14_{-7}	9.53_{-7}	9.54_{-7}	9.17_{-7}	9.05_{-7}
	f	3.12_{-4}	1.73_{-4}	3.49_{-4}	2.92_{-4}	7.13_{-4}	5.77_{-4}	5.51_{-4}	6.33_{-4}

Numerical Experiments

Matrix Completion

Optimization formulation in [Van13]:

$$\min_{X \in \mathbb{R}_r^{m \times n}} f(X) := \frac{1}{2} \| P_\Omega (X - A) \|_F^2,$$

where $\mathbb{R}_r^{m \times n} = \{X \in \mathbb{R}^{m \times n} : \mathrm{rank}(X) = r\}$ is the manifold of fixed-rank matrices, Ω is a given index set, $P_\Omega(A)$ is the given sparse matrix, and $P_\Omega(Z_{i,j}) = Z_{i,j}$ if $(i,j) \in \Omega$; $P_\Omega(Z_{i,j}) = 0$ otherwise,

- Compare with LRBFGS, RCGR [Van13], and LMAFit [WYZ12];

Numerical Experiments

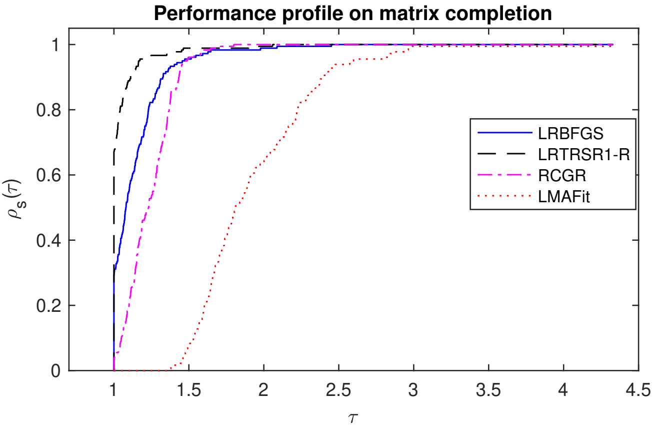
Matrix Completion

Table: An average result of 20 random runs of LRBFGS (i), LRTRSR1-R (ii), RCGR (iii), and LMAFit (iv) with multiple sizes and ranks are reported. *rel* denotes the relative residual $rel = \|P_\Omega(X - A)\|_F / \|P_\Omega(A)$.

		$n = m = 2000$			$n = m = 4000$			$n = m = 8000$		
	r	10	20	30	10	20	30	10	20	30
(i)	*iter*	132	97	101	145	99	93	165	107	93
	nf	159	112	115	172	118	109	195	125	111
	rel	8.11_{-11}	7.13_{-11}	6.67_{-11}	8.71_{-11}	7.30_{-11}	7.10_{-11}	8.59_{-11}	7.69_{-11}	7.16_{-11}
	time	9.47_{-1}	1.50	2.74	1.99	3.63	6.18	5.20	9.09	1.43_{1}
(ii)	*iter*	124	106	96	143	110	105	169	133	112
	nf	125	107	97	144	111	106	170	134	113
	rel	6.13_{-11}	6.66_{-11}	7.53_{-11}	7.30_{-11}	6.60_{-11}	6.53_{-11}	6.96_{-11}	6.66_{-11}	7.51_{-11}
	time	7.18_{-1}	1.41	2.29	1.62	3.39	6.26	4.77	9.76	1.45_{1}
(iii)	*iter*	80	62	57	89	66	61	101	72	64
	nf	81	63	58	90	67	62	102	73	65
	rel	9.17_{-11}	8.46_{-11}	7.94_{-11}	9.34_{-11}	8.92_{-11}	8.19_{-11}	9.51_{-11}	9.09_{-11}	8.49_{-11}
	time	9.06_{-1}	1.55	2.40	2.02	4.35	7.02	5.96	1.10_{1}	1.81_{1}
(iv)	*iter*	338	197	164	402	228	185	491	252	206
	nf	339	198	165	403	229	186	492	253	207
	rel	9.61_{-11}	9.42_{-11}	9.05_{-11}	9.68_{-11}	9.44_{-11}	9.19_{-11}	9.72_{-11}	9.41_{-11}	9.31_{-11}
	time	1.54	2.24	3.46	3.82	5.94	9.62	1.02_{1}	1.89_{1}	2.43_{1}

Numerical Experiments

Matrix Completion



Numerical Experiments

Joint Diagonalization

Joint diagonalization problem [TCA09] is

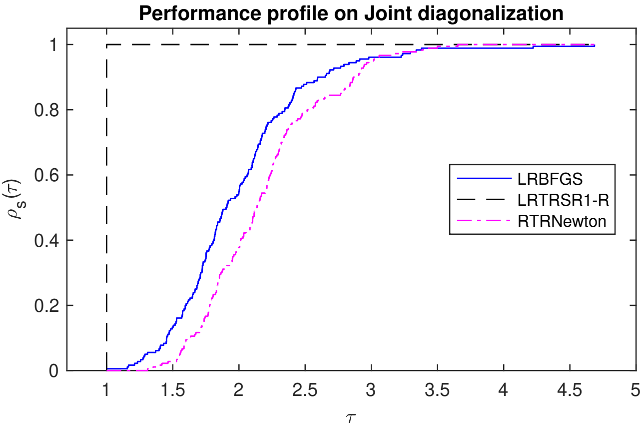
$$f : \mathrm{St}(p, n) \to \mathbb{R} : X \mapsto -\sum_{i=1}^{N} \left\| \mathrm{diag}(X^T C_i X) \right\|_2^2,$$

where $C_1, \ldots, C_N$ are given symmetric matrices, $\mathrm{diag}(M)$ denotes the vector formed by the diagonal entries of M.

- Compare with LRBFGS and RTRNewton [TCA09]

Numerical Experiments

Joint Diagonalization



Numerical Experiments

Phase Retrieval

The phase retrieval problem in [CLS16] is

$$\min_{x \in \mathbb{C}^n} f(x) := \frac{1}{\|b\|_2^2} \|b - \mathrm{diag}(Zxx^* Z^*)\|_2^2,$$

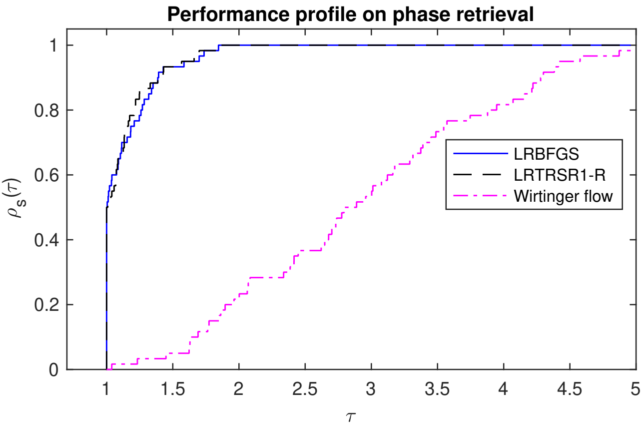
where $b \in \mathbb{R}^m$ is the observation measurements, $Z \in \mathbb{C}^{m \times n}$ is

$$Z = \begin{bmatrix} (\mathcal{F}_s \otimes \mathcal{F}_{s-1} \otimes \dots \mathcal{F}_1)\, \mathrm{ddiag}(w_1) \\ \vdots \\ (\mathcal{F}_s \otimes \mathcal{F}_{s-1} \otimes \dots \mathcal{F}_1)\, \mathrm{ddiag}(w_l) \end{bmatrix},$$

$w_i \in \mathbb{C}^n, i = 1, \dots, l$ denotes masks, $\mathcal{F}_i \in \mathbb{C}^{n_i \times n_i}, i = 1, \dots, s$ denotes the one-dimensional DFT, $\otimes$ denotes the Kronecker product, $m = ln$, and $\mathrm{ddiag}(v)$ denotes a diagonal matrix whose diagonal entries are v.

Numerical Experiments

Phase Retrieval



Numerical Experiments

Conclusion

- Smooth optimization on Riemannian manifold
- Limited-memory symmetric rank-one trust-region subproblem
- Intrinsic representation of tangent vectors
- Global convergence analysis
- Numerical experiments

Thank you

Thank you!

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